QUASIPARTICLES AND VORTICES IN THE HIGH TEMPERATURE CUPRATE SUPERCONDUCTORS

by

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Abstract

In this thesis I present a study of the interplay of vortices and the fermionic quasiparticle states in a quasi 2-dimensional d-wave superconductors, such as high temperature cuprate superconductors. In the first part, I start by analyzing the quasiparticle states in the presence of the magnetic field induced vortex lattice. Unlike in the s-wave superconductor, there are no vortex bound states, rather all the quasiparticle states are extended. By using general arguments based on symmetry principles and by direct (numerical) computation, I show that these extended quasiparticle states are gapped. In addition, they are characterized by a topological invariant with regard to their spin Hall conduction. Finally, I relate the spin Hall conductivity tensor σ_{xy}^{spin} to the thermal Hall conductivity κ_{xy} by deriving the appropriate "Wiedemann-Franz" law. In the second part, I analyze the fluctuations around an ordered d-wave superconductor (in the absence of the external magnetic field), focusing on the interaction between the low energy nodal fermions and the vortex-antivortex phase excitations. It is shown that the corresponding low energy effective theory for the nodal fermions in the normal (non-superconducting) state is QED_3 or quantum electrodynamics in 2+1 space time dimensions. The massless U(1) gauge field encodes the topological interaction between the quasiparticles and the $hc/2e$ vortices at long wavelengths. I analyze the symmetries, correlations and stability of this state. In particular, I study the role of Dirac cone anisotropy and residual interactions in QED_3 as well as the physical meaning of the chiral symmetry breaking. Remarkably, the spin density wave corresponds to an instability of a phase fluctuating d-wave superconductor.
Advisor: Zlatko B. Tesanovic.

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Chapter 1

Introduction

The properties of matter at low temperature amplify nature's most fascinating and least comprehensible laws: the laws of quantum mechanics. This amplification process occurs via a principle of symmetry breaking: when the available phase space is sufficiently restricted, interacting systems with a macroscopically large number of degrees of freedom tend to seek a highly organized state. In other words, the low energy state of a many particle system possesses less symmetry than the laws which are obeyed by its constituents. A commonplace example are solids. Despite the fact that the laws of quantum mechanics are invariant under spatial translation, the atoms in a solid occupy a regular array which is only disrupted by small amplitude vibrations and occasional dislocation defects. It is the translational symmetry that is broken by the solid state of matter. This symmetry breaking has an important ramification: the emergence of Goldstone modes, or long lived degrees of freedom, which tend to restore the symmetry. In the case of a solid, this is embodied by the emergence of transverse elastic waves. In addition, symmetry breaking generally leads to stable topological defects; in solids these are lattice dislocations and disclinations. While such topological defects are quite rare deep inside an ordered state, they can be crucial in destroying the order near a phase transition.

A less commonplace example of symmetry breaking is a superconductor. While the laws of quantum mechanics are invariant under a local U(1) transformation, i.e. multiplication of the wavefunction by a space dependent single valued phase factor, the superconducting state is not. In a superconductor the many body wavefunction acquires rigidity under a phase twist. This leads to most of the fascinating properties of superconductors such as the passage of current without any voltage drop, the Meissner effect, or even the magnetic levitation. The density of low energy fermionic excitations is generically appreciably reduced compared to the non-superconducting state. The Goldstone mode that accompanies the symmetry breaking corresponds to smooth (longitudinal) space-time variations of the phase of the many body wavefunction. On the other hand, 2π phase twists of the wavefunction, or vortices, are the analog of the dislocations in the solid. They form the topological defects in the type-II superconductors. At a phase transition, temperature $T = T_c$, such a superconductor looses its phase rigidity due to the free motion of vortex defects. The temperature interval around T_c in which this physics applies is typically barely detectable in the conventional low T_c superconductors, but there is growing experimental evidence that in the high T_c cuprate superconductors (HTS) vortex unbinding is the dominant mechanism of the loss of long range order. In particular, the underdoped materials appear to exhibit many characteristics associated with vortex physics, in some cases all the way down to zero temperature! Such observations suggest that the quantum analog of vortex unbinding occurs at the superconductor-insulator quantum phase transition.

The underlying theme in this work is the interplay of vortices and the fermionic quasiparticles, the single particle excitations of the superconductor. I start by briefly reviewing the phenomenology of HTS. In Chapter 2, I analyze the quasiparticle states in the presence of the magnetic field induced vortex lattice in a planar d-wave superconductor. Several interesting and non-trivial properties of such a state are derived. In Chapter 3, I then go on to analyze the nature of the fermionic excitations in a quantum phase disordered superconductor. It is shown that quantum unbinding of $hc/2e$ vortex defects produces gauge interactions between quasiparticles. The low energy effective field theory for such a state is argued to be QED_3 (2+1 dimensional electrodynamics with massless fermions). This theory has a very rich structure: it has a phase characterized by very strong interaction between fermions and the gauge field, but in which no symmetries are broken. In addition it also has a broken symmetry phase in which the fermions gain mass. Remarkably, this state can be associated with a spin density wave, a state adiabatically connected to an antiferromagnet, the parent state of all cuprate superconductors.

1.1 Phenomenology of High Temperature Cuprate Superconductors

High temperature cuprate superconductors were discovered by Bednorz and Müller in 1986 [1]. Soon thereafter, Anderson pointed out three essential features of the new materials [2]. First, they are quasi 2-dimensional with weakly coupled CuO_2 planes (Fig. 1.1). Second, the superconductor is created by doping a Mott insulator, and third, Anderson proposed that the combination of proximity to a Mott insulator and low dimensionality would cause the doped material to exhibit fundamentally new behavior, impossible to explain by conventional paradigms of metal physics [2, 3].

In a Mott insulator, charge transport is prohibited not merely by the Pauli exclusion principle (as in a band insulator), but also by strong electron-electron repulsion that pins the electrons to the lattice sites. This insulating state is stable when there is exactly one electron per each site in the valence band, since motion of electrons requires energetically expensive double occupancy. Virtual charge excitations generate a "super-exchange" interaction that favors antiferromagnetic alignment of spins (Fig. 1.1).

Upon doping, the system becomes a weak conductor and eventually a superconductor, Fig. (1.2). However, the cuprates are the only Mott insulators which become superconducting when doped [3]. It was established soon after the discovery of HTS that the superconductivity is due to the electrons forming Cooper pairs (see Fig. 1.3). However, unlike in conventional s-wave superconductors, the gap function changes sign upon 90 degree rotation i.e. it has $d_{x^2-y^2}$ symmetry. As such, it vanishes at four points on the Fermi surface, leading to gapless Fermi points. These points were identified by angle–resolved photoemission spectroscopy (ARPES) (Fig. 1.4). In addition, several ingenious phase sensitive tests, based on Josephson tunneling, were

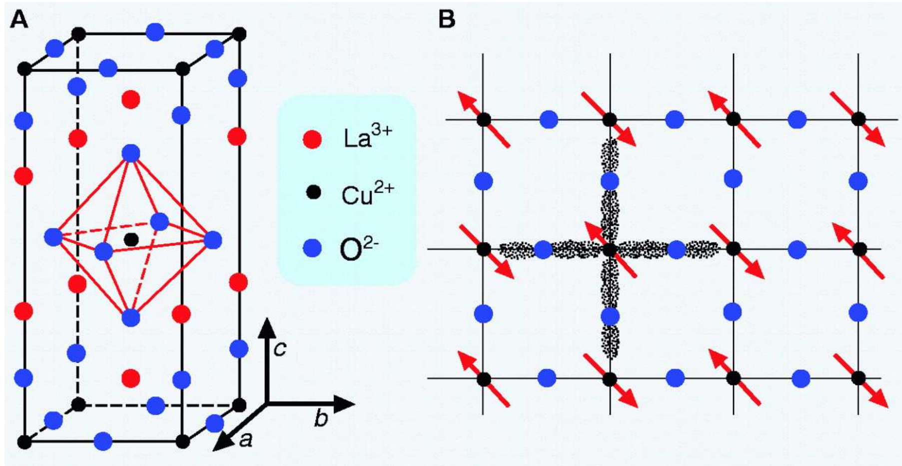


Figure 1.1: (A) The unit cell of $La_{2-x}Sr_xCuO_4$ family of high temperature superconductors. It is believed that most of the interesting physics happens in the Cu-O_2 plane, which extends in the $a-b$ directions. The $c-$axis electronic coupling is very small. In this family of materials, doping is achieved by replacing some of the La atoms for Sr, or by adding interstitial oxygen atoms. This crystalline structure is slightly modified in different high-T_c cuprates, but all share the weakly coupled Cu-O_2 planes. (B) Arrows indicate spin alignment in the antiferromagentic state, the parent state of HTS. (Taken from [3])

developed to demonstrate the change of sign of the pairing amplitude (for a review see [5]). The low energy properties of such d-wave superconductors are governed by the excitations around the four nodal points. These correspond to nodal BCS quasiparticles whose existence was demonstrated by several techniques, but most directly by ARPES. These low energy BCS quasiparticles are responsible, for example, for the linear in temperature depletion of superfluid density [6] or "universal" longitudinal thermal conductivity, which was demonstrated experimentally [7] to depend only on the ratio of the quasiparticle velocity perpendicular and parallel to the Fermi surface at the Fermi points.

The boundary in the phase diagram between the antiferromagnet and the superconductor corresponds to a fascinating phase: the pseudogap state. This state is not a superconductor, but spectroscopically it is nearly indistinguishable from the superconductor as there is a suppression of single particle states at the Fermi surface as well as the nodal points. While the spin fluctuations are gapped, the in-plane charge

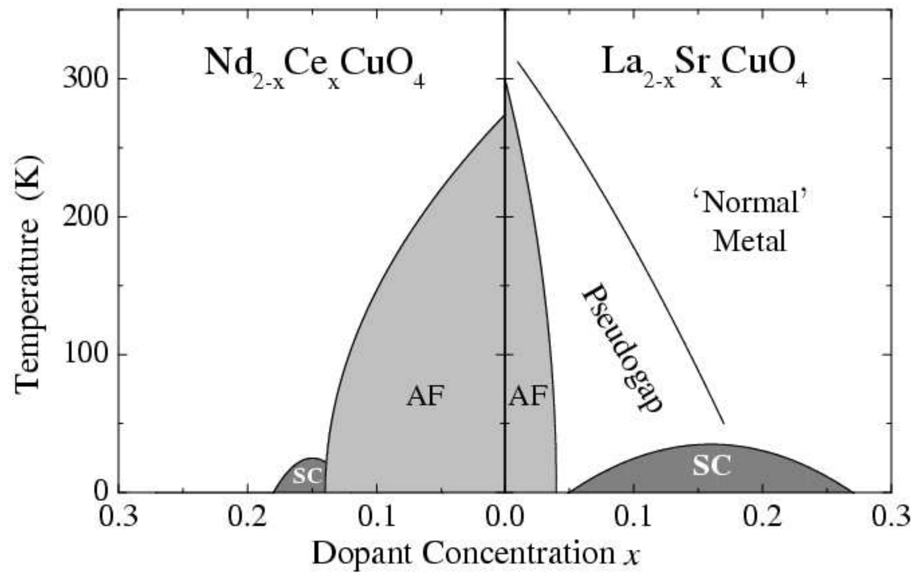


Figure 1.2: Phase diagram of electron and hole doped High Temperature superconductors showing the superconducting (SC), antiferromagnetic (AF), pseudogap and metallic phases. (Taken from [4]).

transport seems unaffected, whereas the c-axis transport is suppressed. Moreover, strong superconducting fluctuations seem to be prominent in this state. Especially striking are the recent observations of anomalously large Nernst signal [8] above T_c in single crystal underdoped cuprates (see Fig. 1.5). In this measurement, a magnetic field is applied perpendicular to the CuO_2 planes, along with a weak thermal gradients within the plane. One then measures the electrical voltage drop perpendicular to the thermal current flow. The signal seen is nearly three orders of magnitude larger than in conventional metals, not unlike in vortex liquid state. However, it is observed several tens on Kelvin above T_c! This suggests that the physics of pseudogap is dominated by strong pairing fluctuations.

In what follows, I shall first study the BCS quasiparticles of a d-wave superconductor with magnetic field induced array of Abrikosov vortices. I shall assume that we are at $T = 0$ far away from any phase transition so that fluctuations can be neglected (e.g. the region between optimally doped and slightly overdoped in the phase diagram (1.2)). Later, I shall concentrate on the pseudogap region, and assume that it is dominated by phase fluctuations, in particular that the pseudogap region is nothing

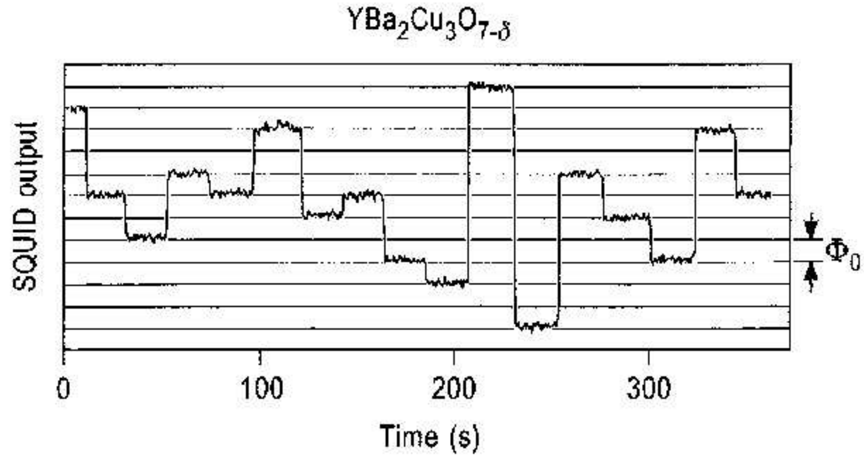
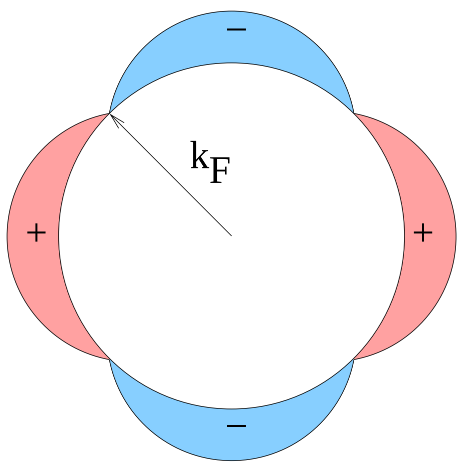


Figure 1.3: The magnetic flux threading through a polycrystalline YBCO ring, monitored with a SQUID magnetometer as a function of time. The flux jumps occur in integral multiples of the superconducting flux quantum $\Phi_0 = hc/2e$. This experiment demonstrates that the superconductivity in high temperature cuprates is due to Cooper pairing. (taken from [5])

but a phase disordered d-wave superconductor.



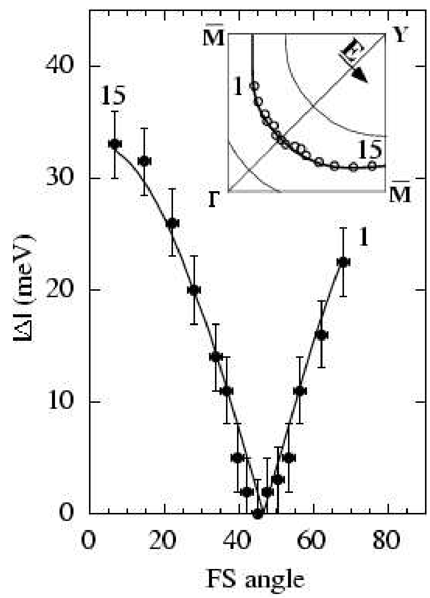


Figure 1.4: The pairing gap of HTS has $d_{x^2-y^2}$-wave symmetry (left). The points on the Fermi surface at which the gap disappears (the nodal points) can be identified in the angle resolved photoemission spectra (right) [4].

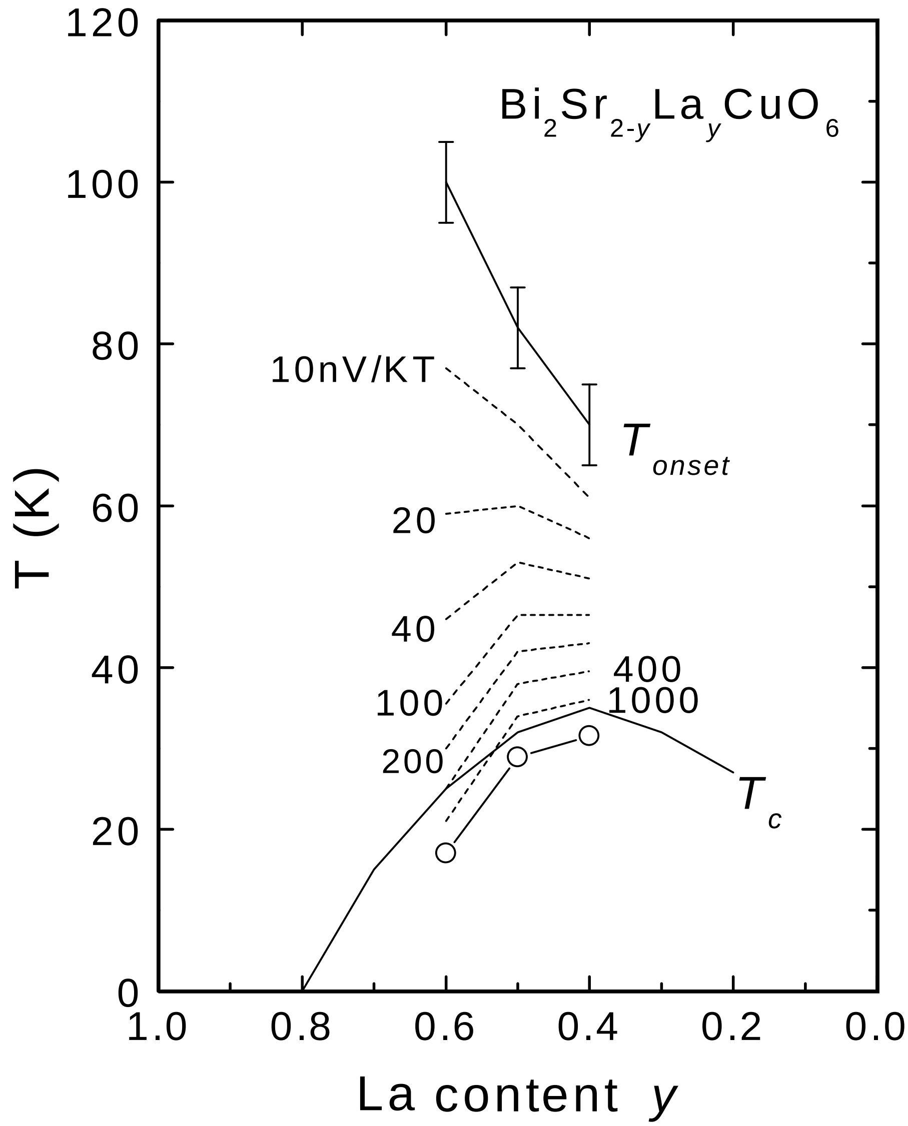


Figure 1.5: Contours of the constant Nernst coefficient in the Temperature vs. doping phase diagram. The units of the Nernst coefficient are nanoVolt per Kelvin per Tesla. At low temperature and in the strongly underdoped region the Nernst signal is almost three orders of magnitude greater than it would be in a normal metal. Such a large signal is typically observed in the vortex liquid state, where it is associated with Josephson phase-slips produced across the sample due to thermally diffusing vortices. In this experiment, however, this signal is seen several tens of Kelvins above the superconducting transition temperature T_c! [8]

Chapter 2

Quasiparticles in the mixed state of HTS

2.1 Introduction

In this section we give a brief review of the fermionic quasiparticle excitations in the superconductors, both conventional and unconventional.

In conventional (s-wave) superconductors the effective attraction between electrons is mainly due to exchange of phonons [9]. The evidence of the phonon mediated interaction comes from the isotope effect i.e. the shift of the transition temperature upon the replacement of the crystal ions with their isotopes. The electrons pair in the lowest angular momentum channel, $l = 0$ or s-wave, and the pairing amplitude does not vary appreciably around the Fermi surface. As a result, the single particle fermionic excitations (quasiparticles) are fully gapped everywhere on the Fermi surface and the quasiparticle density of states vanishes below a specific energy. This has profound consequences for the traditional phenomenology of superconductors. The gap in the fermionic spectrum leads to the well known activated form of the quasiparticle contribution to various thermodynamic and transport properties, and can be directly observed in tunneling spectroscopy (see e.g. [5]). Furthermore, even beyond the mean-field theory, the presence of the pairing gap in the superconducting state cuts off the infra-red singularities which allows perturbative treatment of various types of quasiparticle interactions.

High temperature cuprate superconductors (HTS), however, are different. While the origin of pairing remains an unsolved problem, the cuprates appear to be accurately described by the $d_{x^2-y^2}$-wave order parameter [10], i.e. the pairing happens in $l = 2$ channel whose degeneracy is further split by the crystal field in favor of $d_{x^2-y^2}$. As a result, quasiparticle excitations occur at arbitrarily low energy near the nodal points. These low energy fermionic excitations appear to govern much of the thermodynamics and transport in the HTS materials. This represents a new intellectual challenge [11]: one must devise methods that can incorporate the low energy fermionic excitations into the phenomenology of superconductors, both within the mean-field BCS-like theory and beyond.

This challenge is not trivial and has many diverse components. Low energy quasiparticles are scattered by impurities in novel and unusual ways, depending on the low energy density of states [12]. They interact with external perturbations in ways not encountered in conventional superconductors, and these interactions give rise to new phenomena [13, 14]. The low energy quasiparticles are thus expected to qualitatively affect the quantum critical behavior of HTS (see Chapter 3). Among the many aspects of this new quasiparticle phenomenology a particularly prominent role is played by the low lying quasiparticle excitations in the mixed (or vortex) state [15]. All HTS are extreme type-II systems and have a huge mixed phase extending from the lower critical field H_{c1} which is in the range of 10-100 Gauss to the upper critical field H_{c2} which can be as large as 100-200 T. In this large region the interactions between quasiparticles and vortices play the essential role in defining the nature of thermodynamic and transport properties.

Thermodynamic and transport properties are expected to be rather different for distinct classes of unconventional superconductors. The difference stems from a complex motion of the quasiparticles under the combined effects of both the magnetic field $\mathbf{B}$ and the local drift produced by chiral supercurrents of the vortex state. For example, in HTS the $d_{x^2-y^2}$-wave nature of the gap function results in its vanishing along nodal directions. Along these nodal directions the pair-breaking induced by supercurrents has a particularly strong effect. On the other hand, in unconventional superconductors with the $p_{x\pm iy}$ pairing, Sr_2RuO_4 being a possible candidate [16], the spectrum is fully gapped but the order parameter is chiral even in the absence of external magnetic field. This leads to two different types of vortices for two different field orientations [17, 18].

Still, in all these different situations, the quantum dynamics of quasiparticles in the vortex state contains two essential common ingredients. *First*, there is the purely classical effect of the Doppler shift [13, 14]: quasiparticles' energy is shifted by a locally drifting superfluid, $E(\mathbf{k}) \rightarrow E(\mathbf{k}) - \hbar\mathbf{v}_s(\mathbf{r}) \cdot \mathbf{k}$, where $\mathbf{v}_s(\mathbf{r})$ is the local superfluid velocity. $\mathbf{v}_s(\mathbf{r})$ contains information about vortex configurations, allowing us to connect quasiparticle spectral properties to various cooperative phenomena in the system of vortices [19, 20, 21]. The Doppler shift effect is not peculiar to the vortex state. It also occurs in the Meissner phase [14] and is generally present whenever a quasiparticle experiences a locally uniform drift in the superfluid velocity. *Second*, there is also a purely quantum effect which is intimately tied to the vortex state: as a quasiparticle circles around a vortex while maintaining its quantum coherence, the accumulated phase through a Doppler shift is $\pm\pi$. This implies that there must be an *additional* compensating $\pm\pi$ contribution to the phase on top of the one due to the Doppler shift in order for the wavefunction to remain single-valued [22]. The required $\pm\pi$ contribution is supplied by a "Berry phase" effect and can be built in at the Hamiltonian level as a half-flux Aharonov-Bohm scattering of quasiparticles by vortices [22]. This interplay between the classical (Doppler shift) and purely quantum effect ("Berry phase") is what makes the problem of quasiparticle-vortex interaction particularly fascinating.

Let us briefly review what is already known about the quasiparticles in the vortex state. The initial theoretical investigations of gapped and gapless superconductors in the vortex state were directed along rather separate lines. The low energy quasiparticle spectrum of an s-wave superconductor in the mixed state was originally studied by Caroli, de Gennes and Matricon (CdGM) [23] within the framework of the Bogoliubov-de Gennes equations [24]. Their solution yields well known bound states in the vortex cores. These states are *localized* in the core and have an exponential envelope, the scale of which is set by the BCS coherence length. The low energy end of the spectrum is given by $\epsilon_\mu \sim \mu(\Delta_0^2/E_F)$, where $\mu = 1/2, 3/2, \ldots$, Δ_0 is the overall BCS gap, and E_F is the Fermi energy. This solution can be relatively straightforwardly generalized to a fully *gapped*, *chiral* p-wave superconductor. In this case the low energy quasiparticle spectrum also displays bound vortex core states, whose energy quantization is, however, modified relative to its s-wave counterpart, precisely because of the chiral character of a $p_{x\pm iy}$-wave superconductor and the ensuing shift in the angular momentum. For example, the low energy spectrum of quasiparticles in the singly quantized vortex of the $p_{x\pm iy}$-wave superconductor, possesses a state at exactly zero energy [17, 18].

By comparison, the spectrum of a *gapless* d-wave superconductor in the mixed phase has become the subject of an active debate only relatively recently, fueled by the interest in HTS. Naturally, the first question that arises is what is the analog of the CdGM solution for a single vortex. It is important to realize here that the situation in a $d_{x^2-y^2}$ superconductor is *qualitatively different* from the classic s-wave case [25]: when the pairing state has a finite angular momentum and is not a global eigenstate of the angular momentum L_z (a $d_{x^2-y^2}$ superconductor is an equal admixture of $L_z = \pm 2$ states), the problem of fermionic excitations in the core *cannot* be reduced to a collection of decoupled 1D dimensional eigenvalue equations for each angular momentum channel, the key feature of the CdGM solution. Instead, all channels remain coupled and one must solve a *full* 2D problem. The fully self-consistent numerical solution of the BdG equations [25, 26] reveals the most important physical consequence of this qualitatively new situation: the vortex core quasiparticle states in a pure $d_{x^2-y^2}$ superconductor are *delocalized* with wave functions extended along the nodal directions. The low lying states have a continuous spectrum and, in a broad range of parameters, do not seem to exhibit strong resonant behavior. This is in sharp contrast with a discrete spectrum and true bound quasiparticle states of the CdGM s-wave solution.

A particularly important issue in this context is the nature of the quasiparticle excitations at very low fields, in the presence of a vortex lattice. This is a novel challenge since the spectrum starts as gapless at zero field and at issue is the interaction of these low lying quasiparticles with the vortex lattice. This problem has been addressed via numerical solution of the tight binding model [27], a numerical diagonalization of the continuum model [28] and a semiclassical analysis [13]. Gorkov, Schrieffer [29] and, in a somewhat different context, Anderson [61], predicted that the quasiparticle spectrum is described by a Dirac-like Landau quantization of energy levels

$$E_n = \pm\hbar\omega_H\sqrt{n}, \quad n = 0, 1, ... \tag{2.1}$$

where $\omega_H = \sqrt{2\omega_c\Delta_0/\hbar}$, $\omega_c = eB/mc$ is the cyclotron frequency and Δ_0 is the maximum superconducting gap. The Dirac-like spectrum of Landau levels arises from the linear dispersion of nodal quasiparticles at zero field. This argument neglects the effect of spatially varying supercurrents in the vortex array which were shown to strongly mix individual Landau levels [31].

Recently, Franz and Tešanović (FT) [22] pointed out that the low energy quasiparticle states of a $d_{x^2-y^2}$-wave superconductor in the vortex state are most naturally described by strongly dispersive Bloch waves. This conclusion was based on the particular choice of a singular gauge transformation, which allows for the treatment of the uniform external magnetic field and the effects produced by chiral supercurrents on equal footing. The starting point was the Bogoliubov-de Gennes (BdG) equation linearized around a Dirac node. By employing the singular gauge transformation FT mapped the original problem onto that of a Dirac Hamiltonian in periodic vector and scalar potentials, comprised of an array of an effective magnetic Aharonov-Bohm half-fluxes, and with a vanishing overall magnetic flux per unit cell. The FT gauge transformation allows use of standard band structure and other zero-field techniques to study the quasiparticle dynamics in the presence of vortex arrays, ordered or disordered. Its utility was illustrated in Ref. [22] through computation of the quasiparticle spectra of a square vortex lattice. A remarkable feature of these spectra is the persistence of the massless Dirac node at finite fields and the appearance of the "lines of nodes" in the gap at large values of the anisotropy ratio $\alpha_D = v_F/v_\Delta$, starting at $\alpha_D \simeq 15$. Furthermore, the FT transformation directly reveals that a quasiparticle moving coherently through a vortex array experiences not only a Doppler shift caused by circulating supercurrents but also an *additional*, "Berry phase" effect: the latter is a purely quantum mechanical phenomenon and is absent from a typical semiclassical approach. Interestingly, the cyclotron motion in Dirac cones is *entirely* caused by such "Berry phase" effect, which takes the form of a half-flux Aharonov-Bohm scattering of quasiparticles by vortices, and does *not* explicitly involve the external magnetic field. It is for this reason that the Dirac-like Landau level quantization is absent from the exact quasiparticle spectrum.

Further progress was achieved by Marinelli, Halperin and Simon [32] who presented a detailed perturbative analysis of the linearized Hamiltonian of Ref. [22]. They showed that the presence of the particle-hole symmetry is of key importance in determining the nature of the spectrum of low energy excitations. If the vortices are arranged in a Bravais lattice, they showed that, to all orders in perturbation theory, the Dirac node is preserved at finite fields, i.e the quasiparticle spectrum remains gapless at the Γ point. This result masks intense mixing of individual basis vectors (in the case of Ref. [32] these are Dirac plane waves), including strong mixing of states far removed in energy. The continuing presence of the massless Dirac node at the Γ point after the application of the external field is thus not due to the lack of scattering which is actually remarkably strong. Rather, it is dictated by symmetry: Marinelli *et al.* demonstrated that the crucial agent responsible for the presence of the Dirac node is the particle-hole symmetry, present at every point in the Brillouin zone. The fact that it is the particle-hole symmetry rather than the lack of scattering that protects the Dirac node is clearly revealed in the related problem of a Schrödinger electron in the presence of a single Aharonov-Bohm half-flux, where the density of states acquires a δ function depletion at $\mathbf{k} = 0$ [33], thus shifting part of the spectral weight to infinity due to remarkably strong scattering. The authors of Ref. [32] also corrected Ref. [22] by showing that the "lines of nodes" must actually be the "lines of near nodes" since true zeroes of the energy away from Dirac node are prohibited on symmetry grounds. Still, these "lines" will act as true nodes in all realistic circumstances, due to extraordinarily small excitation energies.

In this work I extend the original analysis which was based solely on the *continuum* description by introducing a tight binding "regularization" of the full lattice BdG Hamiltonian, to which we then apply the FT gauge transformation. The lattice formulation allows us to study what, if any, role is played by *internodal* scattering which is simply not a part of the linearized description. This is important and necessary since the straightforward linearization of BdG equations drops curvature terms and results in the thermal Hall conductivity: $\kappa_{xy} = 0$[15, 34]. We employ the Franz and Tešanović (FT) transformation so that we can use the familiar Bloch representation of the translation group in which the overall chirality of the problem vanishes. This should be contrasted with the original problem where the overall chirality is finite and the magnetic translation group states must be used instead. Naively, it might appear that after an FT singular gauge transformation the effects of the magnetic field have somehow been transformed away since the new problem is found to have zero average effective magnetic field. Of course, this is not true. The presence of magnetic field in the original problem reveals itself fully in the FT transformed quasiparticle wavefunctions. Alternatively, there is an "intrinsic" chirality imposed on the system which cannot be transformed away by a choice of the basis. One manifestation of this chirality is the Hall effect. The utility of the singular gauge transformation in the calculation of the electrical Hall conductivity in the *normal* 2D electron gas in a (non-uniform) magnetic field was realized by Nielsen and Hedegård [35]. They demonstrated that using singularly gauge transformed wavefunctions one still obtains the correct result, giving the electrical Hall conductance quantized in units of e^2/h if the chemical potential lies in the energy gap. In a superconductor, the question of Hall response becomes rather interesting as there is a strong mixing between particles and holes. Evidently, the electrical Hall response is very different from the normal state, since charge is not conserved in the state with broken U(1) symmetry. Therefore, as pointed out in Ref. [36], charge cannot be transported by diffusion. On the other hand, the spin is still a good quantum number[36] and it is natural to ask what is the spin Hall conductivity in the vortex state of an extreme type-II superconductor [37]. Moreover, every channel of spin conduction simultaneously transports entropy [36, 38, 39] and we would expect some variation of the Wiedemann-Franz law to hold between spin and thermal conductivity.

As one of our main results, we derive the Wiedemann-Franz law connecting the spin and thermal Hall transport in the *vortex state* of a d-wave superconductor. In the process, we show that the spin Hall conductivity, σ^s_{xy}, just like the electrical Hall conductivity of a normal state in a magnetic field, is topological in nature and can be explicitly evaluated as a first Chern number characterizing the eigenstates of our singularly gauge transformed problem [37, 40, 41]. Consequently, as $T \to 0$, the spin Hall conductivity is quantized in the units of $\hbar/8\pi$ when the energy spectrum is gapped, which, combined with the Wiedemann-Franz law, implies the quantization of κ_{xy}/T. We then explicitly compute the quantized values of σ^s_{xy} for a sequence of gapped states using our lattice d-wave superconductor model in the case of an inversion-symmetric vortex lattice. Within this model one is naturally led to consider the *BCS-Hofstadter* problem: the BCS pairing problem defined on a uniformly frustrated tight-binding lattice. We find a sequence of plateau transitions, separating gapped states characterized by different quantized values of σ^s_{xy}. At a plateau transition, level crossings take place and σ^s_{xy} changes by an even integer [42]. Both the origin of the gaps in quasiparticle spectra and the sequence of values for σ^s_{xy} are rather different than in the normal state, i.e. in the standard Hofstadter problem [43]. In a superconductor, the gaps are strongly affected by the pairing and the interactions of quasiparticles with a vortex array. The sequence of σ^s_{xy} changes as a function of the pairing strength (and therefore interactions), measured by the maximum value of the gap function Δ [44].

2.2 Quasiparticle excitation spectrum of a d-wave superconductor in the mixed state

The experimental evidence points towards well defined d-wave quasiparticles in cuprate superconductors in the absence of the external magnetic field. This suggests that to zeroth order fluctuations can be ignored and that one can think in terms of an effective BCS Hamiltonian, the simplest of which is written on the 2-D tight-binding lattice with the nearest neighbor interaction thus naturally implementing $d_{x^2-y^2}$ pairing. In question is then the response of such a superconductor to an externally applied magnetic field $\mathbf{B}$. All high temperature superconductors are extreme type-II forming a vortex state in a wide range of magnetic fields. This immediately sets up the contrast between $\mathbf{B} = 0$ and $\mathbf{B} \neq 0$ situations: first, the problem is not spatially uniform and therefore momentum is not a good quantum number and second, the array of $\frac{hc}{2e}$ vortex fluxes poses topological constraint on the quasi-particles encircling the vortices. Therefore, despite ignoring any fluctuations, the problem is far from trivial and demands careful examination.

The natural starting point is therefore the mean-field BCS Hamiltonian written in second quantized form [45]:

$$H = \int d\mathbf{x}\, \psi^\dagger_\alpha(\mathbf{x}) \left(\frac{1}{2m^*}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 - \mu \right) \psi_\alpha(\mathbf{x}) + \int d\mathbf{x} \int d\mathbf{y} [\Delta(\mathbf{x},\mathbf{y}) \psi^\dagger_\uparrow(\mathbf{x}) \psi^\dagger_\downarrow(\mathbf{y}) + \Delta^*(\mathbf{x},\mathbf{y}) \psi_\downarrow(\mathbf{y}) \psi_\uparrow(\mathbf{x})] \quad (2.2)$$

where $\mathbf{A}(\mathbf{x})$ is the vector potential associated with the uniform external magnetic field $\mathbf{B}$, single electron energy is measured relative to the chemical potential μ, $\psi_\alpha(\mathbf{x})$ is the fermion field operator with spin index α, and $\Delta(\mathbf{x},\mathbf{y})$ is the pairing field. For convenience we will define an integral operator $\hat{\Delta}$ such that:

$$\hat{\Delta}\psi(\mathbf{x}) = \int d\mathbf{y} \Delta(\mathbf{x},\mathbf{y}) \psi(\mathbf{y}). \quad (2.3)$$

In the strictest sense, on the mean field level this problem must be solved self-consistently which renders any analytical solution virtually intractable. On the other hand, in the case at hand the vortex lattice is dilute for a wide range of magnetic fields, and by the very nature of cuprate superconductors having short coherence length, the size of the vortex core can be ignored relative to the distance between the vortices. Thus, to the first approximation, all essential physics is captured by fixing the amplitude of the order parameter Δ while allowing vortex defects in its phase. Moreover, on a tight-binding lattice the vortex flux is concentrated inside the plaquette and thus the length-scale associated with the core is implicitly the lattice spacing $\boldsymbol{\delta}$ of the underlying tight-binding lattice. As shown in Ref.[45], under these approximations the d-wave pairing operator in the vortex state can be written as a differential operator:

$$\hat{\Delta} = \Delta_0 \sum_\delta \eta_{\boldsymbol{\delta}} e^{i\phi(\mathbf{x})/2}\, e^{i\boldsymbol{\delta}\cdot\mathbf{p}}\, e^{i\phi(\mathbf{x})/2}. \quad (2.4)$$

The sums are over nearest neighbors and on the square tight-binding lattice $\boldsymbol{\delta} = \pm\hat{x}, \pm\hat{y}$; the vortex phase fields satisfy $\nabla\times\nabla\phi(\mathbf{x}) = 2\pi\hat{z}\sum_i \delta(\mathbf{x}-\mathbf{x}_i)$ with $\mathbf{x}_i$ denoting the vortex positions and $\delta(\mathbf{x}-\mathbf{x}_i)$ being a 2D Dirac delta function; $\mathbf{p}$ is a momentum operator, and

$$\eta_{\boldsymbol{\delta}} = \begin{cases} 1 & \text{if } \boldsymbol{\delta} = \pm\hat{x} \\ -1 & \text{if } \boldsymbol{\delta} = \pm\hat{y}. \end{cases} \tag{2.5}$$

The operator $\eta_{\boldsymbol{\delta}}$ follows from the d-wave pairing: $\Delta = 2\Delta_0[\cos(k_x\delta_x) - \cos(k_y\delta_y)]$. For notational convenience we will use units where $\hbar = 1$ and return to the conventional units when necessary.

It is straightforward to derive the continuum version of the tight binding lattice operator $\hat{\Delta}$ (see Ref.[45]):

$$\begin{aligned}\hat{\Delta} = \frac{1}{2p_F^2}\{\partial_x, \{\partial_x, \Delta(\mathbf{x})\}\} - \frac{1}{2p_F^2}\{\partial_y, \{\partial_y, \Delta(\mathbf{x})\}\} + \\ + \frac{i}{8p_F^2}\Delta(\mathbf{x})\left[(\partial_x^2\phi) - (\partial_y^2\phi)\right],\end{aligned} \tag{2.6}$$

but for convenience we will keep the lattice definition (2.4) throughout. One can always define continuum as an appropriate limit of the tight-binding lattice theory. With the above definitions, the Hamiltonian (2.2) can now be written in the Nambu formalism as

$$H = \int d\mathbf{x}\ \ \Psi^\dagger(\mathbf{x})\ \hat{H}_0\ \Psi(\mathbf{x}) \tag{2.7}$$

where the Nambu spinor $\Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow)$ and the matrix differential operator

$$\hat{H}_0 = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^* & -\hat{h}^* \end{pmatrix}. \tag{2.8}$$

In the continuum formulation $\hat{h} = \frac{1}{2m^*}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 - \mu$, while on the tight-binding lattice:

$$\hat{h} = -t\sum_\delta e^{i\int_{\mathbf{x}}^{\mathbf{x}+\delta}(\mathbf{p}-\frac{e}{c}\mathbf{A})\cdot d\mathbf{l}} - \mu. \tag{2.9}$$

t is the hopping constant and μ is the chemical potential. The equations of motion of the Nambu fields Ψ are then:

$$i\hbar\dot{\Psi} = [\Psi, H] = \hat{H}_0\Psi. \tag{2.10}$$

2.2.1 Particle-Hole Symmetry

The equations of motion (2.10) for stationary states lead to Bogoliubov-de Gennes equations [45]

$$\hat{H}_0 \Phi_n = \epsilon_n \Phi_n. \tag{2.11}$$

The solution of these coupled differential equations are quasi-particle wavefunctions that are rank two spinors in the Nambu space, $\Phi^T(\mathbf{r}) = (u(\mathbf{r}), v(\mathbf{r}))$. The single particle excitations of the system are completely specified once the quasi-particle wavefunctions are given, and as discussed later, transverse transport coefficients can be computed solely on the basis of Φ's. It is a general symmetry of the BdG equations that if $(u_n(\mathbf{r}), v_n(\mathbf{r}))$ is a solution with energy ϵ_n, then there is always another solution $(-v_n^*(\mathbf{r}), u_n^*(\mathbf{r}))$ with energy $-\epsilon_n$ (see for example Ref. [24]).

In addition, on the tight-binding lattice, if the chemical potential $\mu = 0$ in the above BdG Hamiltonian (2.8), then there is a *particle-hole* symmetry in the following sense: if $(u_n(\mathbf{r}), v_n(\mathbf{r}))$ is a solution with energy ϵ_n, then there is always another solution $e^{i\pi(r_x+r_y)}(u_n(\mathbf{r}), v_n(\mathbf{r}))$ with energy $-\epsilon_n$. Thus we can choose:

$$\begin{pmatrix} u_n^{(-)}(\mathbf{r}) \\ v_n^{(-)}(\mathbf{r}) \end{pmatrix} = e^{i\pi(r_x+r_y)} \begin{pmatrix} u_n^{(+)}(\mathbf{r}) \\ v_n^{(+)}(\mathbf{r}) \end{pmatrix}, \tag{2.12}$$

where + (−) corresponds to a solution with positive (negative) energy eigenvalue. We will refer to this as particle hole transformation $\hat{P}_H$.

2.2.2 Franz-Tešanović Transformation and Translation Symmetry

In order to elucidate another important symmetry of the Hamiltonian (2.8), we follow FT [22, 45] and perform a "bipartite" singular gauge transformation on the Bogoliubov-de Gennes Hamiltonian (2.11),

$$\hat{H}_0 \to U^{-1}\hat{H}_0 U, \quad U = \begin{pmatrix} e^{i\phi_e(\mathbf{r})} & 0 \\ 0 & e^{-i\phi_h(\mathbf{r})} \end{pmatrix}, \tag{2.13}$$

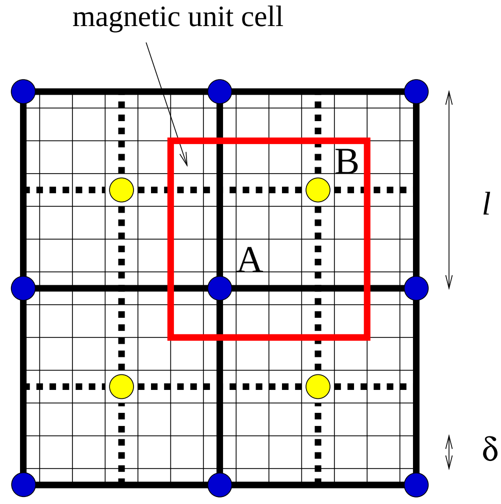


Figure 2.1: Example of A and B sublattices for the square vortex arrangement. The underlying tight-binding lattice, on which the electrons and holes are allowed to move, is also indicated.

where $\phi_e(\mathbf{r})$ and $\phi_h(\mathbf{r})$ are two auxiliary vortex phase functions satisfying

$$\phi_e(\mathbf{r}) + \phi_h(\mathbf{r}) = \phi(\mathbf{r}). \tag{2.14}$$

This transformation eliminates the phase of the order parameter from the pairing term of the Hamiltonian. The phase fields $\phi_e(\mathbf{r})$ and $\phi_h(\mathbf{r})$ can be chosen in a way that avoids multiple valuedness of the wavefunctions. The way to accomplish this is to assign the singular part of the phase field generated by any given vortex to either $\phi_e(\mathbf{r})$ or $\phi_h(\mathbf{r})$, but not both. Physically, a vortex assigned to $\phi_e(\mathbf{r})$ will be seen by electrons and be invisible to holes, while vortex assigned to $\phi_h(\mathbf{r})$ will be seen by holes and be invisible to electrons. For periodic Abrikosov vortex array, we implement the above transformation by dividing vortices into two groups A and B, positioned at $\{\mathbf{r}_i^A\}$ and $\{\mathbf{r}_i^B\}$ respectively (see Fig. 2.1). We then define two phase fields $\phi^A(\mathbf{r})$ and $\phi^B(\mathbf{r})$ such that

$$\nabla \times \nabla \phi^\alpha(\mathbf{r}) = 2\pi \hat{z} \sum_i \delta(\mathbf{r} - \mathbf{r}_i^\alpha), \quad \alpha = A, B, \tag{2.15}$$

and identify $\phi_e = \phi^A$ and $\phi_h = \phi^B$. On the tight-binding lattice the transformed Hamiltonian becomes

$$\hat{H}_N = \sum_{\boldsymbol{\delta}} \left\{ \sigma_3 \left(-t e^{i \int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}} (\mathbf{a} - \sigma_3 \mathbf{v}) \cdot d\mathbf{l}} e^{i \boldsymbol{\delta} \cdot \mathbf{p}} - \mu \right) + \sigma_1 \Delta_0 \eta_{\boldsymbol{\delta}} e^{i \int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}} \mathbf{a} \cdot d\mathbf{l}} e^{i \boldsymbol{\delta} \cdot \mathbf{p}} \right\} \tag{2.16}$$

where

$$\mathbf{v} = \frac{1}{2} \nabla \phi - \frac{e}{c} \mathbf{A}; \quad \mathbf{a} = \frac{1}{2} (\nabla \phi^A - \nabla \phi^B), \tag{2.17}$$

σ_1 and σ_3 are Pauli matrices operating in Nambu space, and the sum is again over the nearest neighbors. Note that the integrand of Eq. (2.16) is proportional to the superfluid velocities

$$\mathbf{v}_s^\alpha = \frac{1}{m^*} (\nabla \phi^\alpha - \frac{e}{c} \mathbf{A}), \quad \alpha = A, B. \tag{2.18}$$

and is therefore explicitly gauge invariant as are the off-diagonal pairing terms.

From the perspective of quasiparticles $\mathbf{v}_s^A$ and $\mathbf{v}_s^B$ can be thought of as *effective* vector potentials acting on electrons and holes respectively. Corresponding effective magnetic field seen by the quasiparticles is $\mathbf{B}_{\text{eff}}^\alpha = -\frac{m^* c}{e} (\nabla \times \mathbf{v}_s^\alpha)$, and can be expressed using Eqs. (2.15) and (2.16) as

$$\mathbf{B}_{\text{eff}}^\alpha = \mathbf{B} - \phi_0 \hat{z} \sum_i \delta(\mathbf{r} - \mathbf{r}_i^\alpha), \quad \alpha = A, B, \tag{2.19}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ is the physical magnetic field and $\phi_0 = hc/e$ is the flux quantum. We observe that quasi-electrons and quasi-holes propagate in the effective field which consists of (almost) uniform physical magnetic field $\mathbf{B}$ and an array of opposing delta function "spikes" of unit fluxes associated with vortex singularities. The latter are different for electrons and holes. As discussed in [22, 45] this choice guarantees that the effective magnetic field vanishes on average, i.e. $\langle \mathbf{B}_{\text{eff}}^\alpha \rangle = 0$ since we have precisely one flux spike (of A and B type) per flux quantum of the physical magnetic field. Flux quantization guarantees that the right hand side of Eq. (2.19) vanishes when averaged over a vortex lattice unit cell containing two physical vortices. It also implies that there must be equal numbers of A and B vortices in the system.

The essential advantage of the choice with vanishing $\langle \mathbf{B}_{\text{eff}}^\alpha \rangle$ is that $\mathbf{v}_s^A$ and $\mathbf{v}_s^B$ can be chosen periodic in space with periodicity of the magnetic unit cell containing an integer number of electronic flux quanta hc/e. Notice that vector potential of a field that does not vanish on average can never be periodic in space. Condition $\langle \mathbf{B}^{\alpha}_{\text{eff}} \rangle = 0$ is therefore crucial in this respect. The singular gauge transformation (2.13) thus maps the original Hamiltonian of fermionic quasiparticles in finite magnetic field onto a new Hamiltonian which is formally in zero average field and has only "neutralized" singular phase windings in the off-diagonal components.

The resulting new Hamiltonian now commutes with translations spanned by the magnetic unit cell i.e.

$$[\hat{T}_{\mathbf{R}}, \hat{H}_N] = 0, \tag{2.20}$$

where the translation operator $\hat{T}_{\mathbf{R}} = \exp(i\mathbf{R}\cdot\mathbf{p})$. We can therefore label eigenstates with a "vortex" crystal momentum quantum number $\mathbf{k}$ and use the familiar Bloch states as the natural basis for the eigen-problem. In particular we seek the eigensolution of the BdG equation $\hat{H}_N\psi = \epsilon\psi$ in the Bloch form

$$\psi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\Phi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} U_{n\mathbf{k}}(\mathbf{r}) \\ V_{n\mathbf{k}}(\mathbf{r}) \end{pmatrix}, \tag{2.21}$$

where $(U_{n\mathbf{k}}, V_{n\mathbf{k}})$ are periodic on the corresponding unit cell, n is a band index and $\mathbf{k}$ is a crystal wave vector. Bloch wavefunction $\Phi_{n\mathbf{k}}(\mathbf{r})$ satisfies the "off-diagonal" Bloch equation $\hat{H}(\mathbf{k})\Phi_{n\mathbf{k}} = \epsilon_{n\mathbf{k}}\Phi_{n\mathbf{k}}$ with the Hamiltonian of the form

$$\hat{H}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{H}_N e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2.22}$$

Note, that the dependence on $\mathbf{k}$, which is bounded to lie in the first Brillouin zone, is continuous. This will become important when topological properties of spin transport are discussed.

2.2.3 Vortex Lattice Inversion Symmetry and Level Crossing

General features of the quasi-particle spectrum can be understood on the basis of symmetry alone. Since the time-reversal symmetry is broken, the Bogoliubov-de Gennes Hamiltonian H_0 (2.11) must be, in general, complex. According to the "non-crossing" theorem of von Neumann and Wigner [46], a complex Hamiltonian can have degenerate eigenvalues unrelated to symmetry only if there are at least three parameters which can be varied simultaneously.

Since the system is two dimensional, with the vortices arranged on the lattice, there are two parameters in the Hamiltonian $\hat{H}(\mathbf{k})$ (2.22): vortex crystal momenta k_x and k_y which vary in the first Brillouin zone. Therefore, we should not expect any degeneracy to occur, *in general*, unless there is some symmetry which protects it. Away from half-filling ($\mu \neq 0$) and with unspecified arrangement of vortices in the magnetic unit cell there is not enough symmetry to cause degeneracy. There is only *global* Bogoliubov-de Gennes symmetry relating quasi-particle energy $\epsilon_{\mathbf{k}}$ at some point $\mathbf{k}$ in the first Brillouin zone to $-\epsilon_{-\mathbf{k}}$.

In order for every quasiparticle band to be either completely below or completely above the Fermi energy, it is sufficient for the vortex lattice to have inversion symmetry. This can be readily seen by the following argument: Consider a vortex lattice with inversion symmetry. Then, by the very nature of the superconducting vortex carrying $\frac{hc}{2e}$ flux, there must be even number of vortices per magnetic unit cell and we are then free to choose Franz-Tesanovic labels A and B in such a way that $\mathbf{v}^A(-\mathbf{r}) = -\mathbf{v}^B(\mathbf{r})$. To see this note that the explicit form of the superfluid velocities can be written as [45]:

$$\mathbf{v}_s^\alpha(\mathbf{r}) = \frac{2\pi\hbar}{m^*}\int \frac{d^2k}{(2\pi)^2}\frac{i\mathbf{k}\times\hat{z}}{k^2}\sum_i e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_i^\alpha)}, \tag{2.23}$$

where $\alpha = A$ or B and $\mathbf{r}_i^\alpha$ denotes the position of the vortex with label α. If the vortex lattice has inversion symmetry then for every $\mathbf{r}_i^A$ there is a corresponding $-\mathbf{r}_i^B$ such that $\mathbf{r}_i^A = -\mathbf{r}_i^B$. Therefore, under space inversion $\mathcal{I}$

$$\mathcal{I}\mathbf{v}^A(\mathbf{r}) = \mathbf{v}^A(-\mathbf{r}) = -\mathbf{v}^B(\mathbf{r}). \tag{2.24}$$

Recall that the tight-binding lattice Bogoliubov-de Gennes Hamiltonian written in the Bloch basis (2.22) reads:

$$\hat{H}(\mathbf{k}) = \sum_{\boldsymbol{\delta}}\left\{\sigma_3\big(-te^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}(\mathbf{a}-\sigma_3\mathbf{v})\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot(\mathbf{k}+\mathbf{p})} - \mu\big) + \sigma_1\Delta_0\eta_{\boldsymbol{\delta}}e^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}\mathbf{a}\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot(\mathbf{k}+\mathbf{p})}\right\} \tag{2.25}$$

where

$$\mathbf{v}(\mathbf{r}) \equiv \frac{1}{2}\big(\mathbf{v}^A(\mathbf{r}) + \mathbf{v}^B(\mathbf{r})\big);\quad \mathbf{a}(\mathbf{r}) \equiv \frac{1}{2}\big(\mathbf{v}^A(\mathbf{r}) - \mathbf{v}^B(\mathbf{r})\big). \tag{2.26}$$

As before σ_1 and σ_3 are Pauli matrices operating in Nambu space and the sum is again over the nearest neighbors. It can be easily seen that upon applying the space inversion $\mathcal{I}$ to $\hat{H}(\mathbf{k})$ followed by complex conjugation $\mathcal{C}$ and $i\sigma_2$ we have a symmetry that for every $\epsilon_{\mathbf{k}}$ there is $-\epsilon_{\mathbf{k}}$, that is:

$$-i\sigma_2 \mathcal{C} I \; \hat{H}(\mathbf{k}) \, \mathcal{I} C i\sigma_2 = -\hat{H}(\mathbf{k}) \tag{2.27}$$

which holds for every point in the Brillouin zone. Therefore, in order for the spectrum *not* to be gapped, we would need band crossing at the Fermi level. But by the non-crossing theorem this cannot happen *in general*. Thus, the quasi-particle spectrum of an inversion symmetric vortex lattice is gapped, unless an external parameter other than k_x and k_y is fine-tuned.

As will be established in the next section, gapped quasi-particle spectrum implies quantization of the transverse spin conductivity σ^s_{xy} as well as κ_{xy}/T for T sufficiently low. Precisely at half-filling ($\mu = 0$) σ^s_{xy} must vanish on the basis of particle-hole symmetry (see the next section). We can then vary the chemical potential so that $\mu \neq 0$ and break particle-hole symmetry. Hence, the chemical potential μ can serve as the third parameter necessary for creating the accidental degeneracy, i.e. at some special values of μ^* the gap at the Fermi level will close (see Fig. 2.2). This results in a possibility of changing the quantized value of σ^s_{xy} by an integer in units of $\hbar/8\pi$ (Fig. 2.2) [42]. By the very nature of the superconducting state, we achieve *plateau* dependence on the chemical potential. This is to be contrasted to the plateaus in the "ordinary" integer quantum Hall effect which are due to the presence of disorder. In our case, the system is clean and the plateaus are due to the magnetic field induced gap and superconducting pairing.

Similarly, we can change the strength of the electron-electron attraction, which is proportional to the maximum value of the superconducting order parameter Δ_0 while keeping the chemical potential μ fixed. Again, as can be seen in Fig. 2.3, at some special values Δ^*_0 the spectrum is gapless and the quantized Hall conductance undergoes a transition.

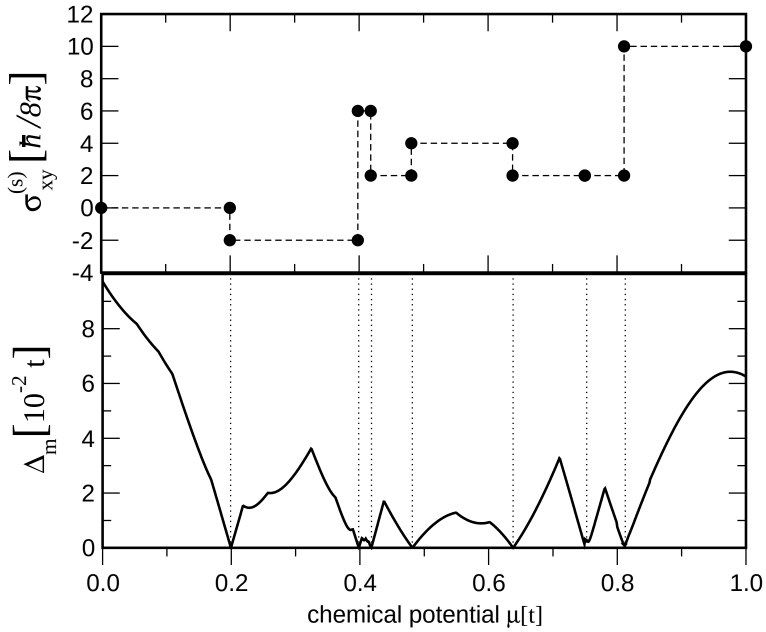


Figure 2.2: The mechanism for changing the quantized spin Hall conductivity is through exchanging the topological quanta via ("accidental") gap closing. The upper panel displays spin Hall conductivity σ^s_{xy} as a function of the chemical potential μ. The lower panel shows the magnetic field induced gap Δ_m in the quasiparticle spectrum. Note that the change in the spin Hall conductivity occurs precisely at those values of chemical potential at which the gap closes. Hence the mechanism behind the changes of σ^s_{xy} is the exchange of the topological quanta at the band crossings. The parameters for the above calculation were: square vortex lattice, magnetic length $l = 4\delta$, $\Delta = 0.1t$ or equivalently the Dirac anisotropy $\alpha_D = 10$.

2.3 Topological quantization of spin and thermal Hall conductivities

Note, that the Hamiltonian in Eq. (2.2) is our starting *unperturbed* Hamiltonian. In order to compute the linear response to externally applied perturbations we will have to add terms to Eq. (2.2). In particular, we will consider two types of perturbations in the later sections: First, partly for theoretical convenience, we will consider a weak gradient of magnetic field ($\nabla\mathbf{B}$) on top of the uniform $\mathbf{B}$ already taken into account fully by Eq. (2.2). The $\nabla\mathbf{B}$ term induces spin current in the superconductor [47]. The response is then characterized by spin conductivity tensor σ^s which

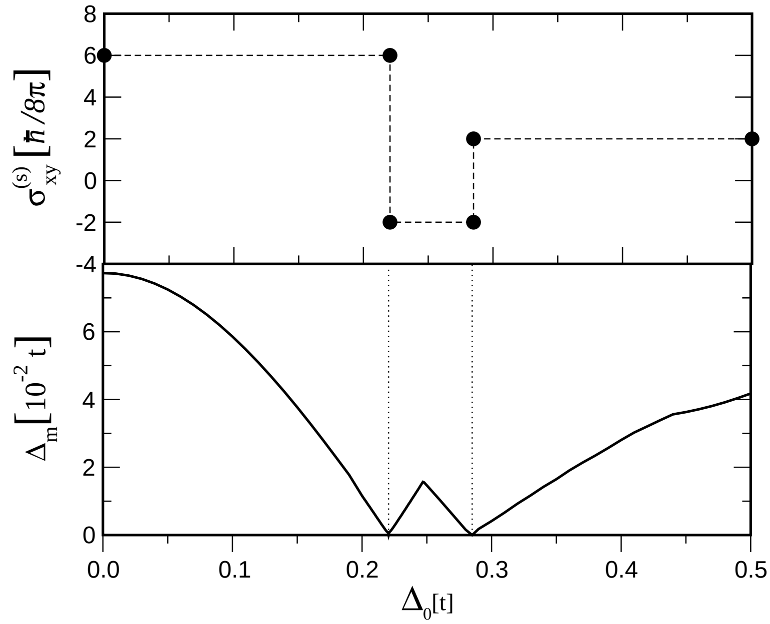


Figure 2.3: The upper panel displays spin Hall conductivity σ^s_{xy} as a function of the maximum superconducting order parameter Δ_0. The lower panel shows the magnetic field induced gap in the quasiparticle spectrum. The change in the spin Hall conductivity occurs at those values of Δ_0 at which the gap closes. The parameters for the above calculation were: square vortex lattice, magnetic length $l = 4\delta$, $\mu = 2.2t$.

in general has non-zero off-diagonal components. Second, we consider perturbing the system by pseudo-gravitational field, which formally induces flow of energy (see [48, 49, 50, 51]) and allows us to compute thermal conductivity κ_{xy} via linear response.

2.3.1 Spin Conductivity

Within the framework of linear-response theory [52], spin dc conductivity can be related to the spin current-current retarded correlation function $D^R_{\mu\nu}$ through :

$$\sigma^s_{\mu\nu} = \lim_{\Omega\to 0} \lim_{q_1,q_2\to 0} -\frac{1}{i\Omega}\Bigg(D^R_{\mu\nu}(q_1,q_2,\Omega) - D^R_{\mu\nu}(q_1,q_2,0)\Bigg). \tag{2.28}$$

The retarded correlation function $D^R_{\mu\nu}(\Omega)$ can in turn be related to the Matsubara finite temperature correlation function

$$D_{\mu\nu}(i\Omega) = -\int_0^\beta e^{i\tau\Omega}\langle T_\tau j^s_\mu(\tau) j^s_\nu(0)\rangle d\tau \tag{2.29}$$

as

$$\lim_{q_1,q_2\to 0} D^R_{\mu\nu}(q_1,q_2,\Omega) = D_{\mu\nu}(i\Omega \to \Omega + i0). \tag{2.30}$$

In the Eq. (2.29) the spatial average of the spin current $\mathbf{j}^s(\tau)$ is implicit, since we are looking for dc response of spatially inhomogeneous system. In the next section we derive the spin current and evaluate the above formulae.

Spin Current

In order to find the dc spin conductivity, we must first find the spin current. More precisely, since we are looking only for the spatial average of the spin current $\mathbf{j}^s(\tau)$ we just need its $\mathbf{k} \to 0$ component. In direct analogy with the $\mathbf{B} = 0$ situation [38], we can define the spin current by the continuity equation:

$$\dot{\rho}^s + \nabla \cdot \mathbf{j}^s = 0 \tag{2.31}$$

where $\rho^s = \frac{\hbar}{2}(\psi^\dagger_{\uparrow x}\psi_{\uparrow x} - \psi^\dagger_{\downarrow x}\psi_{\downarrow x})$ is the spin density projected onto z-axis. We can then use equations of motion for the ψ fields (2.10) and compute the current density $\mathbf{j}^s$ from (2.31).

In the limit of $q \to 0$ the spin current can be written as (see A.2)

$$j^s_\mu = \frac{\hbar}{2}\Psi^\dagger V_\mu \Psi, \tag{2.32}$$

where the Nambu field $\Psi^\dagger = (\psi^\dagger_\uparrow, \psi_\downarrow)$ and the generalized velocity matrix operator V_μ satisfies the following commutator identity

$$V_\mu = \frac{1}{i\hbar}[x_\mu, \hat{H}_0]. \tag{2.33}$$

The equation (2.33) is a direct restatement of the fact that spin can be transported by diffusion, i.e. it is a good quantum number in a superconductor, and that the average velocity of its propagation is just the group velocity of the quantum mechanical wave.

In the clean limit, the transverse spin conductivity σ^s_{xy} defined in Eq. (2.28) is

$$\sigma^s_{xy}(T) = \frac{\hbar^2}{4i}\sum_{m,n}(f_n - f_m)\frac{V^{mn}_y V^{nm}_x}{(\epsilon_n - \epsilon_m + i0)^2}, \tag{2.34}$$

where $f_m = \big(1+\exp(\beta\epsilon_n)\big)^{-1}$ is the Fermi-Dirac distribution function evaluated at energy ϵ_m. For details of the derivation see A.2. The indices m and n label quantum numbers of particular states. The matrix elements V_μ^{mn} are

$$V_\mu^{mn} = \langle m|V_\mu|n\rangle = \int d\mathbf{x}\,(u_m^*, v_m^*)V_\mu \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{2.35}$$

where the particle-hole wavefunctions u_m, v_n satisfy the Bogoliubov-deGennes equation (2.11). Note that unlike the longitudinal dc conductivity, transverse conductivity is well defined even in the absence of impurity scattering. This demonstrates the fact that the transverse conductivity is *not* dissipative in origin. Rather, its nature is topological.

In the limit of $T \to 0$ the expression (2.34) for σ_{xy}^s becomes

$$\sigma_{xy}^s = \frac{\hbar^2}{4i}\sum_{\epsilon_m<0<\epsilon_n}\frac{V_x^{mn}V_y^{nm} - V_y^{mn}V_x^{nm}}{(\epsilon_m-\epsilon_n)^2}. \tag{2.36}$$

The summation extends over all states below and above the Fermi energy which, by the nature of the superconductor, is automatically set to zero.

Vanishing of the Spin Conductivity at Half Filling ($\mu = 0$)

It is useful to contrast the *semiclassical* approach with the full quantum mechanical treatment of transverse spin conductivity. In semiclassical analysis the starting unperturbed Hamiltonian is usually defined in the *absence* of magnetic field $\mathbf{B}$. One then assumes semiclassical dynamics and no inter-band transitions. In this picture, if there is particle-hole symmetry in the original ($\mathbf{B} = 0$) Hamiltonian, then there will be no transverse spin (thermal) transport, since the number of carriers with a given spin (energy) will be the same in opposite directions. In this context, similar argument was put forth in Ref. [15]. However, the problem of a d-wave superconductor is not so straightforward. As pointed out in Ref.[45], in the nodal (Dirac fermion) approximation, the vector potential is solely due to the superflow while the uniform magnetic field enters as a Doppler shift i.e. Dirac scalar potential. Semiclassical analysis must then be started from this vantage point and the above conclusions are not straightforward, since the quasiparticle motion is irreducibly quantum mechanical.

Here we present an argument for the full quantum mechanical problem, without relying on the semiclassical analysis. We show that spin conductivity tensor (2.34) vanishes at $\mu = 0$ due to particle hole symmetry (2.12). First note that the Fermi-Dirac distribution function satisfies $f(\epsilon) = 1 - f(-\epsilon)$. Therefore, the factor $f_m - f_n$ changes sign under the particle-hole transformation $\hat{P}_H$ (2.12) while the denominator $(\epsilon_m - \epsilon_n)^2$ clearly remains unchanged. In addition, each of the matrix elements V_μ^{mn} changes sign under $\hat{P}_H$. Thus the double summation over all states in Eq. (2.34) yields zero.

Consequently the spin transport vanishes for a clean strongly type-II BCS d-wave superconductor on a tight binding lattice at half filling. Due to Wiedemann-Franz law, which we derive in the next section, thermal Hall conductivity also vanishes at half filling at sufficiently low temperatures. Note that this result is independent of the vortex arrangement i.e. it holds even for disordered vortex array and does not rely on any approximation regarding inter- or intra- nodal scattering.

Topological Nature of Spin Hall Conductivity at T=0

In order to elucidate the topological nature of σ^s_{xy}, we make use of the translational symmetry discussed in Section 2.2.2 and formally assume that the vortex arrangement is periodic. However, the detailed nature of the vortex lattice will not be specified and thus any vortex arrangement is allowed within the magnetic unit cell. The conclusions we reach are therefore quite general.

We will first rewrite the velocity matrix elements V_μ^{mn} using the singularly gauge transformed basis as discussed in Section 2.2.2. Inserting unity in the form of the FT gauge transformation (2.13)

$$V_\mu^{mn} = \langle m | V_\mu | n \rangle = \langle m | U\, U^{-1} V_\mu U\, U^{-1} | n \rangle. \tag{2.37}$$

The transformed basis states $U^{-1}|n\rangle$ can now be written in the Bloch form as $e^{i\mathbf{k}\cdot\mathbf{r}}|n_\mathbf{k}\rangle$ and therefore the matrix element becomes

$$V_\mu^{mn} = \langle m_\mathbf{k} | e^{-i\mathbf{k}\cdot\mathbf{r}} U^{-1} V_\mu U e^{i\mathbf{k}\cdot\mathbf{r}} | n_\mathbf{k} \rangle = \langle m_\mathbf{k} | V_\mu(\mathbf{k}) | n_\mathbf{k} \rangle. \tag{2.38}$$

We used the same symbol $\mathbf{k}$ for both bra and ket because the crystal momentum in the first Brillouin zone is conserved. The resulting velocity operator can now be simply expressed as

$$V_\mu(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial \hat{H}(\mathbf{k})}{\partial k_\mu}, \tag{2.39}$$

where $\hat{H}_0(\mathbf{k})$ was defined in (2.22). Furthermore the matrix elements of the partial derivatives of $\hat{H}(\mathbf{k})$ can be simplified according to

$$\langle m_{\mathbf{k}}|\frac{\partial \hat{H}(\mathbf{k})}{\partial k_\mu}|n_{\mathbf{k}}\rangle = (\epsilon^n_{\mathbf{k}} - \epsilon^m_{\mathbf{k}})\langle m_{\mathbf{k}}|\frac{\partial n_{\mathbf{k}}}{\partial k_\mu}\rangle = -(\epsilon^n_{\mathbf{k}} - \epsilon^m_{\mathbf{k}})\langle\frac{\partial m_{\mathbf{k}}}{\partial k_\mu}|n_{\mathbf{k}}\rangle, \tag{2.40}$$

for $m \neq n$. Utilizing Eqs. (2.39) and (2.40), Eq. (2.36) for σ^s_{xy} can now be written as

$$\sigma^s_{xy} = \frac{\hbar}{4i}\int\frac{d\mathbf{k}}{(2\pi)^2}\sum_{\epsilon^m<0<\epsilon^n}\langle\frac{\partial m_{\mathbf{k}}}{\partial k_x}|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|\frac{\partial m_{\mathbf{k}}}{\partial k_y}\rangle - \langle\frac{\partial m_{\mathbf{k}}}{\partial k_y}|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|\frac{\partial m_{\mathbf{k}}}{\partial k_x}\rangle. \tag{2.41}$$

The identity $\sum_{\epsilon^m_{\mathbf{k}}<0<\epsilon^n_{\mathbf{k}}}(|m_{\mathbf{k}}\rangle\langle m_{\mathbf{k}}| + |n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|) = 1$, can be further used to simplify the above expression to read

$$\sigma^{s,m}_{xy} = \frac{\hbar}{8\pi}\frac{1}{2\pi i}\int d\mathbf{k}\left(\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_x}\middle|\frac{\partial m_{\mathbf{k}}}{\partial k_y}\right\rangle - \left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_y}\middle|\frac{\partial m_{\mathbf{k}}}{\partial k_x}\right\rangle\right) \tag{2.42}$$

where $\sigma^{s,m}_{xy}$ is a contribution to the spin Hall conductance from a completely filled band m, well separated from the rest of the spectrum. Therefore the integral extends over the entire magnetic Brillouin zone that is topologically a two-torus T^2. Let us define a vector field $\hat{A}$ in the magnetic Brillouin zone as

$$\hat{A}(\mathbf{k}) = \langle m_{\mathbf{k}}|\nabla_{\mathbf{k}}|m_{\mathbf{k}}\rangle, \tag{2.43}$$

where $\nabla_{\mathbf{k}}$ is a gradient operator in the $\mathbf{k}$ space. From (2.42) this contribution becomes

$$\sigma^{s,m}_{xy} = \frac{\hbar}{8\pi}\frac{1}{2\pi i}\int d\mathbf{k}[\nabla_{\mathbf{k}} \times \hat{A}(\mathbf{k})]_z, \tag{2.44}$$

where $[]_z$ represents the third component of the vector. The topological aspects of the quantity in (2.44) were extensively studied in the context of integer quantum Hall effect (see e.g. [53]) and it is a well known fact that

$$\frac{1}{2\pi i}\int d\mathbf{k}[\nabla_{\mathbf{k}} \times \hat{A}(\mathbf{k})]_z = C_1 \tag{2.45}$$

where C_1 is a first Chern number that is an integer. Therefore, a contribution of each filled band to σ^s_{xy} is

$$\sigma^{s,m}_{xy} = \frac{\hbar}{8\pi} N \tag{2.46}$$

where N is an integer. The assumption that the band must be separated from the rest of the spectrum can be relaxed. If two or more fully filled bands cross each other the sum total of their contributions to spin Hall conductance is quantized even though nothing guarantees the quantization of the individual contributions. The quantization of the total spin Hall conductance requires a gap in the single particle spectrum at the Fermi energy. As discussed in the Section 2.2.3, the general single particle spectrum of the d-wave superconductor in the vortex state with inversion-symmetric vortex lattice is gapped and therefore the quantization of σ^s_{xy} is guaranteed.

2.3.2 Thermal Conductivity

Before discussing the nature of the quasi-particle spectrum, we will establish a Wiedemann-Franz law between spin conductivity and thermal conductivity for a d-wave superconductor. This relation is naturally expected to hold for a very general system in which the quasi-particles form a degenerate assembly i.e. it holds even in the presence of elastically scattering impurities.

Following Luttinger [48], and Smrčka and Středa [50] we introduce a pseudo-gravitational potential $\chi = \mathbf{x} \cdot \mathbf{g}/c^2$ into the Hamiltonian (2.7) where $\mathbf{g}$ is a constant vector. The purpose is to include a coupling to the energy density on the Hamiltonian level. This formal trick allows us to equate statistical ($T\nabla(1/T)$) and mechanical ($\mathbf{g}$) forces so that the thermal current $\mathbf{j}^Q$, in the long wavelength limit given by

$$\mathbf{j}^Q = L^Q(T) \left(T\nabla\frac{1}{T} - \nabla\chi \right), \tag{2.47}$$

will vanish in equilibrium. Therefore it is enough to consider only the dynamical force $\mathbf{g}$ to calculate the phenomenological coefficient $L^Q_{\mu\nu}$. Note that thermal conductivity κ_{xy} is

$$\kappa_{\mu\nu}(T) = \frac{1}{T} L^Q_{\mu\nu}(T). \tag{2.48}$$

When the BCS Hamiltonian H introduced in Eq.(2.2) becomes perturbed by the pseudo-gravitational field, the resulting Hamiltonian H_T has the form

$$H_T = H + F \tag{2.49}$$

where F incorporates the interaction with the perturbing field:

$$F = \frac{1}{2} \int d\mathbf{x} \;\; \Psi^\dagger(\mathbf{x}) \; (\hat{H}_0 \chi + \chi \hat{H}_0) \; \Psi(\mathbf{x}). \tag{2.50}$$

Since χ is a small perturbation, to the first order in χ the Hamiltonian H_T can be written as

$$H_T = \int d\mathbf{x} \;\; (1 + \frac{\chi}{2}) \Psi^\dagger(\mathbf{x}) \; \hat{H}_0 \; (1 + \frac{\chi}{2}) \Psi(\mathbf{x}) \tag{2.51}$$

i.e. the application of the pseudo-gravitational field results in rescaling of the fermion operators:

$$\Psi \rightarrow \tilde{\Psi} = (1 + \frac{\chi}{2}) \Psi. \tag{2.52}$$

If we measure the energy relative to the Fermi level, the transport of heat is equivalent to the transport of energy. In analogy with the Section 2.3.1, we define the heat current $\mathbf{j}^Q$ through diffusion of the energy-density h_T. From conservation of the energy-density the continuity equation follows

$$\dot{h}_T + \nabla \cdot \mathbf{j}^Q = 0. \tag{2.53}$$

In the limit of $q \rightarrow 0$ the thermal current is

$$j^Q_\mu = \frac{i}{2} \left(\tilde{\Psi}^\dagger V_\mu \dot{\tilde{\Psi}} - \dot{\tilde{\Psi}}^\dagger V_\mu \tilde{\Psi} \right). \tag{2.54}$$

For details see A.3. Note that the quantum statistical average of the current has two contributions, both linear in χ,

$$\langle j^Q_\mu \rangle = \langle j^Q_{0\mu} \rangle + \langle j^Q_{1\mu} \rangle \equiv - \, (K^Q_{\mu\nu} + M^Q_{\mu\nu}) \partial_\nu \chi. \tag{2.55}$$

The first term is the usual Kubo contribution to $L^Q_{\mu\nu}$ while the second term is related to magnetization of the sample [54] for transverse components of $\kappa_{\mu\nu}$ and vanishes for the longitudinal components. In A.3 we show that at $T = 0$ the term related to magnetization cancels the Kubo term and therefore the transverse component of $\kappa_{\mu\nu}$ is zero at $T = 0$. To obtain finite temperature response, we perform Sommerfeld expansion and derive Wiedemann-Franz law for spin and thermal Hall conductivity.

As shown in the A.3

$$L^Q_{\mu\nu}(T) = -\left(\frac{2}{\hbar}\right)^2 \int d\xi\, \xi^2 \frac{df(\xi)}{d\xi} \tilde{\sigma}^s_{\mu\nu}(\xi) \tag{2.56}$$

where

$$\tilde{\sigma}^s_{xy}(\xi) = \frac{\hbar^2}{4i} \sum_{\epsilon_m < \xi < \epsilon_n} \frac{V_x^{mn} V_y^{nm} - V_y^{mn} V_x^{nm}}{(\epsilon_m - \epsilon_n)^2}. \tag{2.57}$$

Note that $\tilde{\sigma}^s_{\mu\nu}(\xi = 0) = \sigma^s_{\mu\nu}(T = 0)$. For a superconductor at low temperature the derivative of the Fermi-Dirac distribution function is

$$-\frac{df(\xi)}{d\xi} = \delta(\xi) + \frac{\pi^2}{6}(k_B T)^2 \frac{d^2}{d\xi^2}\delta(\xi) + \cdots \tag{2.58}$$

Substituting (2.58) into (2.56) we obtain

$$L^Q_{\mu\nu}(T) = \frac{4\pi^2}{3\hbar^2}(k_B T)^2 \sigma^s_{\mu\nu}, \tag{2.59}$$

where $\sigma^s_{\mu\nu}$ is evaluated at $T = 0$. Finally, using (2.48), in the limit of $T \to 0$

$$\kappa_{\mu\nu}(T) = \frac{4\pi^2}{3}\left(\frac{k_B}{\hbar}\right)^2 T \sigma^s_{\mu\nu}. \tag{2.60}$$

We recognize the Wiedemann-Franz law for the spin and thermal conductivity in the above equation. As mentioned, this relation is quite general in that it is independent of the spatial arrangement of the vortex array or elastic impurities. Thus, quantization of the transverse spin conductivity σ^s_{xy} implies quantization of κ_{xy}/T in the limit of $T \to 0$.

2.4 Discussion and Conclusion

In conclusion, we examined a general problem of 2D type-II superconductors in the vortex state with inversion symmetric vortex lattice. The single particle excitation spectrum is typically gapped and results in quantization of transverse spin conductivity σ^s_{xy} in units of $\hbar/8\pi$ [42]. The topological nature of this phenomenon is discussed. The size of the magnetic field induced gap Δ_m in unconventional d-wave superconductors is not universal and in principle can be as large as several percent of the maximum superconducting gap Δ_0. By virtue of the Wiedemann-Franz law, which we derive for the d-wave Bogoliubov-de Gennes equation in the vortex state, the thermal conductivity $\kappa_{xy} = \frac{4\pi^2}{3}\left(\frac{k_B}{\hbar}\right)^2 T\sigma^s_{xy}$ as $T \to 0$. Thus at $T \ll \Delta_m$ the quantization of κ_{xy}/T will be observable in clean samples with negligible Lande g factor and with well ordered Abrikosov vortex lattice. In conventional superconductors, the size of Δ_m is given by Matricon-Caroli-deGennes vortex bound states $\sim \Delta^2/\epsilon_F$.

In real experimental situations, Lande g factor is not necessarily small. In fact it is close to 2 in cuprates [55]. The Zeeman effect must therefore be included in the analysis. Furthermore, the Zeeman splitting, wherein the magnetic field acts as a chemical potential for the (spinful) quasiparticles, shifts the levels that are populated below the gap by $\pm\boldsymbol{\mu}_{spin}\mathbf{B} = \pm g\frac{\hbar e}{4mc}B$ or in the tight binding units by $\pm g\pi t\left(\frac{a_0}{l}\right)^2$.

In the absence of any Zeeman effect the spectrum is gapped. Since this (mini)gap Δ_m is associated with the curvature terms in the continuum BdG equation, it is expected to scale as B. Since the Zeeman effect also scales as B the topological quantization studied depends on the magnitude of the coefficients. While this coefficient is known for the Zeeman effect ($g \approx 2$), the exact form of Δ_m is not known. However, quite generally, we expect that Δ_m increases with increasing the maximum zero field gap Δ_0. Therefore we could (at least theoretically) always arrange for the quantization to persist despite the competition from the Zeeman splitting.

Detailed numerical examination of the quasiparticle spectra reveals that the spectrum remains gapped for some parameters even if $g = 2$. Thus, although the Zeeman splitting is a competing effect, in some situations it is not enough to prevent the quantization of σ^s_{xy} and consequently of κ_{xy}/T.

We have explicitly evaluated the quantized values of σ^s_{xy} on the tight-binding lattice model of $d_{x^2-y^2}$-wave BCS superconductor in the vortex state and showed that in principle a wide range of integer values can be obtained. This should be contrasted with the notion that the effect of a magnetic field on a d-wave superconductor is solely to generate a $d + id$ state for the order parameter, as in that case $\sigma^s_{xy} = \pm 2$ in units of $\hbar/8\pi$. In the presence of a vortex lattice, the situation appears to be more complex.

By varying some external parameter, for instance the strength of the electron-electron attraction which is proportional to Δ_0 or the chemical potential μ, the gap closing is achieved i.e. $\Delta_m = 0$. The transition between two different values of σ^s_{xy} occurs precisely when the gap closes and topological quanta are exchanged. The remarkable new feature is the *plateau* dependence of σ^s_{xy} on the Δ_0 or μ. It is qualitatively different from the plateaus in the ordinary integer quantum Hall effect which are essentially due to disorder. In superconductors, the plateaus happen in a clean system because the gap in the quasiparticle spectrum is generated by the superconducting pairing interactions.

Chapter 3

QED_3 theory of the phase disordered d-wave superconductors

3.1 Introduction

We will now turn our attention to the fluctuations around the broken symmetry state. As is well known, the fluctuations of the superconducting order parameter, a U(1) field, are crucial in describing the critical behavior below the upper critical dimension $d_u = 4$, and destroy the long range order altogether at and below the lower critical dimension $d_l = 2$. This can be seen by assuming the ordered state and computing the correlations between two fields separated by large distance. In 2 dimensions the correlations vanish algebraically at low temperatures due to the presence of a soft phase mode, thereby destroying the (true) long range order. In 3 dimensions the long range order persist for a finite range of temperatures.

Generically, we would expect that when the measured superconducting coherence length is long, then the mean-field description should be accurate. However, when the coherence length is short, we expect the fluctuations to play a significant role. In cuprate superconductors, the coherence length is very short $\approx 10\mathring{A}$, and moreover, they are highly anisotropic layered materials, which suggests that there are strong fluctuations of the order parameter.

The question we shall try to address in this chapter is what is the nature of the fermionic excitations in a fluctuating d-wave superconductor. Inspired by experiments on the underdoped cuprates [8, 57], we shall assume that the long range order is destroyed by phase fluctuations while the amplitude of the order parameter remains finite. Such a state, without being a superconductor, would appear to have a d-wave-like gap: a pseudogap. We shall not try to give a very detailed account of the origin of the phase fluctuations, rather we shall simply assume, phenomenologically, presence or absence of the long range order. The d-wave superconductor, with well defined BCS quasiparticles, then serves as a point of departure.

We thus assume that the most important effect of strong correlations at the basic microscopic level is to form a large pseudogap of d-wave symmetry, which is predominantly pairing in origin, i.e. arises form the particle-particle (p-p) channel. We can then study the renormalization group fate of various residual interactions among the BCS quasiparticles, only to discover that short range interactions are irrelevant by power counting and thus do not affect the d-wave pseudogap fixed point. Thus, if only short range interactions are taken into account, the low energy BCS quasiparticles remain well defined.

However, there are important additional interactions: BCS quasiparticles couple to the collective modes of the pairing pseudogap, in particular they couple to the (soft) phase mode. We analyze this coupling to show that while the fluctuations in the longitudinal mode at T=0 are not sufficient to destroy the long lived low energy BCS quasiparticles, the transverse fluctuations (vortices) do introduce a novel topological frustration to quasiparticle propagation. We shall argue that this frustration can be encoded by a low energy effective field theory which takes the form of (an anisotropic) QED_3 or quantum electrodynamics in 2+1 dimensions [58]. In its symmetric phase, QED_3 is governed by the interacting critical fixed point, with quasiparticles whose lifetime decays algebraically with energy, leading to a non-Fermi liquid behavior for its fermionic excitations. This "algebraic" Fermi liquid (AFL) [58] displaces conventional Fermi liquid as the underlying theory of the pseudogap state.

The AFL (symmetric QED_3) suffers an intrinsic instability when vortex-antivortex fluctuations and residual interactions become *too strong*. The topological frustration is relieved by the spontaneous generation of mass for fermions, while the Berry gauge field remains massless. In the field theory literature on QED_3 this instability is known as the dynamical chiral symmetry breaking (CSB) and is a well-studied and established phenomenon [59], although some uncertainty remains about its more quantitative aspects [60]. In cuprates, the region of such strong vortex fluctuations corresponds to heavily underdoped samples. When reinterpreted in electron coordinates, the CSB leads to spontaneous creation of a whole plethora of nearly degenerate ordered and gapped states from within the AFL. We shall discuss this in Section 3.6. Notably, the manifold of CSB states contains an incommensurate antiferromagnetic insulator (spin-density wave (SDW)). It is a manifestation of a remarkable richness of the QED_3 theory that both the "algebraic" Fermi liquid and the SDW and other CSB insulating states arise from the one and the same starting point.

3.2 Vortex quasiparticle interaction

3.2.1 Protectorate of the pairing pseudogap

The starting point is the assumption that the pseudogap is predominantly particle-particle or pairing (p-p) in origin and that it has a $d_{x^2-y^2}$ symmetry. This assumption is given mathematical expression in the partition function:

$$\begin{aligned} Z &= \int \mathcal{D}\Psi^\dagger(\mathbf{r},\tau) \int \mathcal{D}\Psi(\mathbf{r},\tau) \int \mathcal{D}\varphi(\mathbf{r},\tau) \exp\left[-S\right], \\ S &= \int d\tau \int d^2r \{\Psi^\dagger\partial_\tau\Psi + \Psi^\dagger\mathcal{H}\Psi + (1/g)\Delta^*\Delta\}, \end{aligned} \quad (3.1)$$

where τ is the imaginary time, $\mathbf{r} = (x, y)$, g is an effective coupling constant in the $d_{x^2-y^2}$ channel, and $\Psi^\dagger = (\bar{\psi}_\uparrow, \psi_\downarrow)$ are the standard Grassmann variables representing coherent states of the Bogoliubov-de Gennes (BdG) fermions. The effective Hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{H} = \begin{pmatrix} \hat{\mathcal{H}}_e & \hat{\Delta} \\ \hat{\Delta}^* & -\hat{\mathcal{H}}_e^* \end{pmatrix} + \mathcal{H}_{\rm res} \quad (3.2)$$

with $\hat{\mathcal{H}}_e = \frac{1}{2m}(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A})^2 - \epsilon_F$, $\hat{\mathbf{p}} = -i\nabla$ (we take $\hbar = 1$), and $\hat{\Delta}$ the d-wave pairing operator (see Eq. 2.6),

$$\hat{\Delta} = \frac{1}{k_F^2}\{\hat{p}_x, \{\hat{p}_y, \Delta\}\} - \frac{i}{4k_F^2}\Delta\big(\partial_x\partial_y\varphi\big), \tag{3.3}$$

where $\Delta(\mathbf{r},\tau) = |\Delta|\exp(i\varphi(\mathbf{r},\tau))$ is the center-of-mass gap function and $\{a,b\} \equiv (ab+ba)/2$. As discussed in Chapter 2, the second term in Eq. (3.3) is necessary to preserve the overall gauge invariance. Notice that we have rotated $\hat{\Delta}$ from $\mathrm{d}_{x^2-y^2}$ to d_{xy} to simplify the continuum limit. $\int \mathcal{D}\varphi(\mathbf{r},\tau)$ denotes the integral over smooth ("spin wave") and singular (vortex) phase fluctuations. Amplitude fluctuations of $\hat{\Delta}$ are suppressed at or just below T^* and the amplitude itself is frozen at $2\Delta \sim 3.56T^*$ for $T \ll T^*$.

The fermion fields $\psi_\uparrow$ and $\psi_\downarrow$ appearing in Eqs. (3.1,3.2) do not necessarily refer to the bare electrons. Rather, they represent some *effective* low-energy fermions of the theory, already fully renormalized by high-energy interactions, expected to be strong in cuprates due to Mott-Hubbard correlations near half-filling [61]. The precise structure of such fermionic effective fields follows from a more microscopic theory and is not of our immediate concern here – we are only relying on the *absence* of *true* spin-charge separation which allows us to write the effective pairing term (3.2) in the BCS-like form. The experimental evidence that supports this reasoning, at least within the superconducting state and its immediate vicinity, is rather overwhelming [8, 57, 62, 63]. Furthermore, $\mathcal{H}_{\rm res}$ represents the "residual" interactions, i.e. the part dominated by the effective interactions in the p-h channel. Our main assumption is equivalent to stating that such interactions are in a certain sense "weak" and less important part of the effective Hamiltonian $\mathcal{H}$ than the large pairing interactions already incorporated through $\hat{\Delta}$. This notion of "weakness" of $\mathcal{H}_{\rm res}$ will be defined more precisely and with it the region of validity of our theory.

3.2.2 Topological frustration

The global U(1) gauge invariance mandates that the partition function (3.1) must be independent of the overall choice of phase for $\hat{\Delta}$. We should therefore aim to eliminate the phase $\varphi(\mathbf{r},\tau)$ from the pairing term (3.2) in favor of $\partial_\mu\varphi$ terms [$\mu=(x,y,\tau)$] in the fermionic action. For the regular ("spin-wave") piece of φ this is easily accomplished by absorbing a phase factor $\exp(i\frac{1}{2}\varphi)$ into both spin-up and spin-down fermionic fields. This amounts to "screening" the phase of $\Delta(\mathbf{r},\tau)$ ("XY phase") by a "half-phase" field ("half-XY phase") attached to $\psi_\uparrow$ and $\psi_\downarrow$. However, as discussed in the Chapter 2, when dealing with the singular part of φ, such transformation "screens" physical singly quantized $hc/2e$ superconducting vortices with "half-vortices" in the fermionic fields. Consequently, this "half-angle" gauge transformation must be accompanied by branch cuts in the fermionic fields which originate and terminate at vortex positions and across which the quasiparticle wavefunction must switch its sign. These branch cuts are mathematical manifestation of a fundamental physical effect: in presence of $hc/2e$ vortices the motion of quasiparticles is *topologically frustrated* since the natural elementary flux associated with the quasiparticles is hc/e i.e. twice as large . The physics of this topological frustration is at the origin of all non-trivial dynamics discussed in this work.

Dealing with branch cuts in a fluctuating vortex problem is very difficult due to their non-local character and defeats the original purpose of reducing the problem to that of fermions interacting with *local* fluctuating superflow field, i.e. with $\partial_\mu\varphi$. Instead, in order to avoid the branch cuts, non-locality and non-single valued wavefunctions, we employ the singular gauge transformation introduced in the Chapter 2, hereafter referred to as 'FT' transformation:

$$\bar{\psi}_\uparrow \to \exp(-i\varphi_A)\bar{\psi}_\uparrow, \quad \bar{\psi}_\downarrow \to \exp(-i\varphi_B)\bar{\psi}_\downarrow, \tag{3.4}$$

where $\varphi_A+\varphi_B=\varphi$. Here $\varphi_{A(B)}$ is the singular part of the phase due to $A(B)$ vortex defects:

$$\nabla\times\nabla\varphi_{A(B)} = 2\pi\hat{z}\sum_i q_i\delta\left(\mathbf{r}-\mathbf{r}_i^{A(B)}(\tau)\right), \tag{3.5}$$

with $q_i=\pm1$ denoting the topological charge of i-th vortex and $\mathbf{r}_i^{A(B)}(\tau)$ its position. The labels A and B represent some convenient but otherwise arbitrary division of vortex defects (loops or lines in $\varphi(\mathbf{r},\tau)$) into two sets. The transformation (3.4) "screens" the original superconducting phase φ (or "XY phase") with two ordinary "XY phases"

φ_A and φ_B attached to fermions. Both φ_A and φ_B are simply phase configurations of $\Delta(\mathbf{r},\tau)$ but with fewer vortex defects. The key feature of the transformation (3.4) is that it accomplishes "screening" of the physical $hc/2e$ vortices without introducing any branch cuts into the wavefunctions. The topological frustration still remains, but now it is expressed through local fields. The resulting fermionic part of the action is then:

$$\mathcal{L}' = \bar{\psi}_\uparrow[\partial_\tau + i(\partial_\tau\varphi_A)]\psi_\uparrow + \bar{\psi}_\downarrow[\partial_\tau + i(\partial_\tau\varphi_B)]\psi_\downarrow + \Psi^\dagger\mathcal{H}'\Psi, \tag{3.6}$$

where the transformed Hamiltonian $\mathcal{H}'$ is:

$$\begin{pmatrix} \frac{1}{2m}(\hat{\pi}+\mathbf{v})^2 - \epsilon_F & \hat{D} \\ \hat{D} & -\frac{1}{2m}(\hat{\pi}-\mathbf{v})^2 + \epsilon_F \end{pmatrix},$$

with $\hat{D} = (\Delta_0/2k_F^2)(\hat{\pi}_x\hat{\pi}_y + \hat{\pi}_y\hat{\pi}_x)$ and $\hat{\pi} = \hat{\mathbf{p}} + \mathbf{a}$.

The singular gauge transformation (3.4) generates a "Berry" gauge potential

$$a_\mu = \frac{1}{2}(\partial_\mu\varphi_A - \partial_\mu\varphi_B) \quad , \tag{3.7}$$

which describes half-flux Aharonov-Bohm scattering of quasiparticles on vortices. a_μ couples to BdG fermions *minimally* and mimics the effect of branch cuts in quasiparticle-vortex dynamics. Closed fermion loops in the Feynman path-integral representation of (3.1) acquire the (-1) phase factors due to this half-flux Aharonov-Bohm effect just as they would from a branch cut attached to a vortex defect.

The above (± 1) prefactors of the BdG fermion loops come on top of general and ever-present U(1) phase factors $\exp(i\delta)$, where the phase δ depends on spacetime configuration of vortices. These U(1) phase factors are supplied by the "Doppler" gauge field

$$v_\mu = \frac{1}{2}(\partial_\mu\varphi_A + \partial_\mu\varphi_B) \equiv \frac{1}{2}\partial_\mu\varphi \quad , \tag{3.8}$$

which denotes the classical part of the quasiparticle-vortex interaction. The coupling of v_μ to fermions is the same as that of the usual electromagnetic gauge field A_μ and is therefore *non-minimal*, due to the pairing term in the original Hamiltonian $\mathcal{H}$ (3.2). It is this non-minimal interaction with v_μ, which we call the Meissner coupling, that is responsible for the U(1) phase factors $\exp(i\delta)$. These U(1) phase factors are "random", in the sense that they are not topological in nature – their values depend on detailed distribution of superfluid fields of all vortices and "spin-waves" as well as on the internal structure of BdG fermion loops, i.e. what is the sequence of spin-up and spin-down portions along such loops. In this respect, while its minimal coupling to BdG fermions means that, within the lattice d-wave superconductor model, one is naturally tempted to represent the Berry gauge field a_μ as a *compact* U(1) gauge field, the Doppler gauge field v_μ is decidedly *non-compact*, lattice or no lattice [64]. a_μ and v_μ as defined by Eqs. (3.7,3.8) are *not independent* since the discrete spacetime configurations of vortex defects $\{\mathbf{r}_i(\tau)\}$ serve as *sources* for *both*.

All choices of the sets A and B in transformation (3.4) are completely *equivalent* – different choices represent different singular gauges and v_μ, and therefore $\exp(i\delta)$, are invariant under such transformations. a_μ, on the other hand, changes but only through the introduction of ($\pm$) unit Aharonov-Bohm fluxes at locations of those vortex defects involved in the transformation. Consequently, the Z_2 style (± 1) phase factors associated with a_μ that multiply the fermion loops remain unchanged. In order to symmetrize the partition function with respect to this singular gauge, we define a generalized transformation (3.4) as the sum over all possible choices of A and B, i.e., over the entire family of singular gauge transformations. This is an Ising sum with 2^{N_l} members, where N_l is the total number of vortex defects in $\varphi(\mathbf{r},\tau)$, and is itself yet another choice of the singular gauge. The many benefits of such symmetrized gauge will be discussed shortly but we stress here that its ultimate function is calculational convenience. What actually matters for the physics is that the original φ be split into two XY phases so that the vortex defects of every distinct topological class are apportioned *equally* between φ_A and φ_B (3.4) [58].

The above symmetrization leads to the new partition function: $Z \to \tilde{Z}$:

$$\tilde{Z} = \int \mathcal{D}\tilde{\Psi}^\dagger \int \mathcal{D}\tilde{\Psi} \int \mathcal{D}v_\mu \int \mathcal{D}a_\mu \exp\left[-\int_0^\beta d\tau \int d^2r \tilde{\mathcal{L}}\right] , \qquad (3.9)$$

in which the half-flux-to-minus-half-flux (Z_2) symmetry of the singular gauge transformation (3.4) is now manifest:

$$\tilde{\mathcal{L}} = \tilde{\Psi}^\dagger[(\partial_\tau + ia_\tau)\sigma_0 + iv_\tau\sigma_3]\tilde{\Psi} + \tilde{\Psi}^\dagger\mathcal{H}'\tilde{\Psi} + \mathcal{L}_0[v_\mu, a_\mu], \qquad (3.10)$$

where $\mathcal{L}_0$ is the "Jacobian" of the transformation given by

$$e^{-\int_0^\beta d\tau \int d^2r \mathcal{L}_0} = 2^{-N_l} \sum_{A,B} \int \mathcal{D}\varphi(\mathbf{r},\tau) \tag{3.11}$$
$$\times\delta[v_\mu - \tfrac{1}{2}(\partial_\mu\varphi_A + \partial_\mu\varphi_B)]\delta[a_\mu - \tfrac{1}{2}(\partial_\mu\varphi_A - \partial_\mu\varphi_B)].$$

Here σ_μ are the Pauli matrices, $\beta = 1/T$, $\mathcal{H}'$ is given in Eq. (3.6) and, for later convenience, $\mathcal{L}_0$ will also include the energetics of vortex core overlap driven by amplitude fluctuations and thus independent of long range superflow (and of $\mathbf{A}$). We call the transformed quasiparticles $\tilde{\Psi}^\dagger = (\bar{\tilde{\psi}}_\uparrow, \tilde{\psi}_\downarrow)$ appearing in (3.10) "topological fermions" (TF). TF are the natural fermionic excitations of the pseudogapped normal state. They are electrically neutral and are related to the original quasiparticles by the inversion of transformation (3.4).

By recasting the original problem in terms of topological fermions we have accomplished our original goal: the interactions between quasiparticles and vortices are now described solely in terms of two local superflow fields [58]:

$$v_{A\mu} = \partial_\mu\varphi_A\ ; \qquad v_{B\mu} = \partial_\mu\varphi_B\ , \tag{3.12}$$

which we can think of as superfluid velocities associated with the phase configurations $\varphi_{A(B)}(\mathbf{r},\tau)$ of a (2+1)-dimensional XY model with periodic boundary conditions along the τ-axis. Our Doppler and Berry gauge fields v_μ and a_μ are linear combinations of $v_{A\mu}$ and $v_{B\mu}$. Note that a_μ is produced *exclusively* by vortex defects since the "spin-wave" configurations of φ can be fully absorbed into v_μ. All that remains is to perform the sum in (3.9) over all the "spin-wave" fluctuations and all the spacetime configurations of vortex defects $\{\mathbf{r}_i(\tau)\}$ of this (2+1)-dimensional XY model.

3.2.3 "Coarse-grained" Doppler and Berry U(1) gauge fields

The exact integration over the phase $\varphi(\mathbf{r},\tau)$ is prohibitively difficult. To proceed by analytic means we devise an approximate procedure to integrate over vortex-antivortex positions $\{\mathbf{r}_i(\tau)\}$ in (3.9) which will capture qualitative features of the long-distance, low-energy physics of the original problem. A hint of how to devise such an approximation comes from examining the role of the Doppler gauge field v_μ

in the physics of this problem. For simplicity, we consider the finite T case where we can ignore the τ-dependence of $\varphi(\mathbf{r},\tau)$. The results are easily generalized, with appropriate modifications, to include quantum fluctuations.

We start by noting that $\mathbf{V}_s = 2\mathbf{v} - (2e/c)\mathbf{A}$ is just the physical superfluid velocity [65], invariant under both $A \leftrightarrow B$ singular gauge transformations (3.4) and ordinary electromagnetic U(1) gauge symmetry. The superfluid velocity (Doppler) field, swirling around each (anti)vortex defect, is responsible for the vast majority of phenomena that are associated with vortices: long range interactions between vortex defects, coupling to external magnetic field and the Abrikosov lattice of the mixed state, Kosterlitz-Thouless transition, etc. Remarkably, its role is essential even for the physics discussed here, although it now occupies a supporting role relative to the Berry gauge field a_μ. Note that if $\mathbf{a}$ (a_μ) were absent – then, upon integration over topological fermions in (3.10), we obtain the following term in the effective Lagrangian:

$$M^2\left(\nabla\varphi - \frac{2e}{c}\mathbf{A}\right)^2 + (\cdots) \quad , \tag{3.13}$$

where $(\cdots)$ denotes higher order powers and derivatives of $2\nabla\varphi - (2e/c)\mathbf{A}$. In the above we have replaced $\mathbf{v} \to (1/2)\nabla\varphi$ to emphasize that the leading term, with the coefficient M^2 proportional to the bare superfluid density, is just the standard superfluid-velocity-squared term of the continuum XY model – the notation M^2 for the coefficient will become clear in a moment. We can now write $\nabla\varphi = \nabla\varphi_{\rm vortex} + \nabla\varphi_{\rm spin-wave}$ and re-express the transverse portion of (3.13) in terms of (anti)vortex positions $\{\mathbf{r}_i\}$ to obtain a familiar form:

$$\to \frac{M^2}{2\pi}\sum_{\langle i,j\rangle} \ln|\mathbf{r}_i - \mathbf{r}_j| \tag{3.14}$$

or, by using $\nabla\cdot\nabla\varphi_{\rm vortex} = 0$ and $\nabla\times\nabla\varphi_{\rm vortex} = 2\pi\rho(\mathbf{r})$, equivalently as:

$$\to \frac{M^2}{2\pi}\int d^2r \int d^2r' \rho(\mathbf{r})\rho(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'| \quad , \tag{3.15}$$

where $\rho(\mathbf{r}) = 2\pi\sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i)$ is the vortex density.

The Meissner coupling of $\mathbf{v}$ to fermions is very strong – it leads to familiar long range interactions between vortices which constrain vortex fluctuations to a remarkable degree. To make this statement mathematically explicit we introduce the Fourier transform of the vortex density $\rho(\mathbf{q}) = \sum_i \exp(i\mathbf{q}\cdot\mathbf{r}_i)$ and observe that its variance satisfies:

$$\langle \rho(\mathbf{q})\rho(-\mathbf{q})\rangle \propto q^2/M^2 \quad . \tag{3.16}$$

Vortex defects form an *incompressible* liquid – the long distance vorticity fluctuations are strongly suppressed. This "incompressibility constraint" is naturally enforced in (3.9) by replacing the integral over *discrete* vortex positions $\{\mathbf{r}_i\}$ with the integral over *continuously* distributed field $\bar{\rho}(\mathbf{r})$ with $\langle\bar{\rho}(\mathbf{r})\rangle = 0$. The Kosterlitz-Thouless transition and other vortex phenomenology are still maintained in the non-trivial structure of $\mathcal{L}_0[\bar{\rho}(\mathbf{r})]$. But the long wavelength form of (3.13) now reads:

$$M^2\left(2\bar{\mathbf{v}} - \frac{2e}{c}\mathbf{A}\right)^2 + (\cdots) \quad , \tag{3.17}$$

and can be interpreted as a *massive* action for a U(1) gauge field $\bar{\mathbf{v}}$. The latter is the *coarse grained* Doppler gauge field defined by

$$\nabla\times\bar{\mathbf{v}}(\mathbf{r}) = \pi\bar{\rho}(\mathbf{r}) \quad . \tag{3.18}$$

The coarse-graining procedure has made $\mathbf{v}$ into a massive U(1) gauge field $\bar{\mathbf{v}}$ whose influence on TF disappears in the long wavelength limit. We can therefore drop it from low energy fermiology.

The full problem also contains the Berry gauge field $\mathbf{a}$ (a_μ) which now must be restored. However, having replaced the Doppler field $\mathbf{v}$ (defined by discrete vortex sources) with the *distributed* quantity (3.18), we *cannot* simply continue to keep $\mathbf{a}$ (a_μ) specified by half-flux Dirac (Aharonov-Bohm) strings located at *discrete* vortex positions $\{\mathbf{r}_i\}$. Instead, $\mathbf{a}$ (a_μ) must be replaced by a new gauge field which reflects the "coarse-graining" that has been applied to $\mathbf{v}$ – simply put, the Z_2-valued Berry gauge field (3.7) of the original problem (3.10) must be *"dressed"* in such a way as to *best compensate* for the error introduced by "coarse-graining" $\mathbf{v}$.

Note that if we insist on replacing $\mathbf{v}$ by its "coarse-grained" form (3.18), the only way to achieve this is to "coarse-grain" *both* $\mathbf{v}_A$ and $\mathbf{v}_B$ in the same manner:

$$\nabla\times\mathbf{v}_A \to 2\pi\rho_A \quad ; \quad \nabla\times\mathbf{v}_B \to 2\pi\rho_B \quad , \tag{3.19}$$

where $\rho_{A,B}(\mathbf{r})$ are now continuously distributed densities of $A(B)$ vortex defects (the reader should contrast this with (3.7,3.8)). This is because the *elementary* vortex variables of our problem are *not* the sources of $\mathbf{v}$ and $\mathbf{a}$; rather, they are the sources of $\mathbf{v}_A$ and $\mathbf{v}_B$. We cannot separately fluctuate or "coarse-grain" the Doppler and the Berry "halves" of a given vortex defect – they are *permanently confined* into a physical ($hc/2e$) vortex. On the other hand, we *can* independently fluctuate A and B vortices – this is why it was important to use the singular gauge (3.4) to rewrite the original problem solely in terms of A and B vortices and associated superflow fields (3.12).

We can now reassemble the coarse-grained Doppler and Berry gauge fields as:

$$\mathbf{v} = \frac{1}{2}(\mathbf{v}_A + \mathbf{v}_B) \;\; ; \;\; \mathbf{a} = \frac{1}{2}(\mathbf{v}_A - \mathbf{v}_B) \;\; , \tag{3.20}$$

with the straightforward generalization to (2+1)D:

$$v = \frac{1}{2}(v_A + v_B) \;\; ; \;\; a = \frac{1}{2}(v_A - v_B) \;\; . \tag{3.21}$$

The coupling of $\mathbf{v}_A(v_A)$ and $\mathbf{v}_B(v_B)$ to fermions is a hybrid of Meissner and minimal coupling [64]. They contribute a product of U(1) phase factors $\exp(i\delta_A) \times \exp(i\delta_B)$ to the BdG fermion loops with both $\exp(i\delta_A)$ and $\exp(i\delta_B)$ being "random" in the sense of previous subsection. Upon coarse-graining $\mathbf{v}_A(v_A)$ and $\mathbf{v}_B(v_B)$ turn into non-compact U(1) gauge fields and therefore $\mathbf{v}(v)$ and $\mathbf{a}(a)$ must as well.

3.2.4 Further remarks on FT gauge

The above "coarse-grained" theory must have the following symmetry: it has to be invariant under the exchange of spin-up and spin-down labels, $\psi_\uparrow \leftrightarrow \psi_\downarrow$, *without* any changes in $\mathbf{v}$ (v). This symmetry insures that (an arbitrarily preselected) S_z component of the *spin*, which is the same for TF and real electrons, decouples from the physical superfluid velocity which naturally must couple only to *charge*. When dealing with discrete vortex defects this symmetry is guaranteed by the singular gauge symmetry defined by the family of transformations (3.4). However, if we replace $\mathbf{v}_A$ and $\mathbf{v}_B$ by their distributed "coarse-grained" versions, the said symmetry is preserved only in the FT gauge. This is seen by considering the effective Lagrangian expressed in terms of coarse-grained quantities:

$$\mathcal{L} \;\to\; \tilde{\Psi}^\dagger \partial_\tau \tilde{\Psi} + \tilde{\Psi}^\dagger \mathcal{H}' \tilde{\Psi} + \mathcal{L}_0[\rho_A, \rho_B] \quad , \tag{3.22}$$

where $\mathcal{H}'$ is given by (3.6), $\mathbf{v}_A$, $\mathbf{v}_B$ are connected to ρ_A, ρ_B via (3.19), and $\mathcal{L}_0$ is independent of τ.

The problem lurks in $\mathcal{L}_0[\rho_A, \rho_B]$ – this is just the entropy functional of fluctuating *free* $A(B)$ vortex-antivortex defects and has the following symmetry: $\rho_{A(B)} \to -\rho_{A(B)}$ with $\rho_{B(A)}$ kept unchanged. This symmetry reflects the fact that the entropic "interactions" do not depend on vorticity. Above the Kosterlitz-Thouless transition we can expand:

$$\mathcal{L}_0 \to \frac{K_A}{4}(\nabla \times \mathbf{v}_A)^2 + \frac{K_B}{4}(\nabla \times \mathbf{v}_B)^2 + (\cdots), \tag{3.23}$$

where the ellipsis denote higher order terms and the coefficients $K^{-1}_{A(B)} \to n_l^{A(B)}$ (see Appendix B.1 for details).

The above discussed symmetry of Hamiltonian $\mathcal{H}'$ (3.6) demands that $\mathbf{v}$ and $\mathbf{a}$ (3.20,3.21) be the natural choice for independent distributed vortex fluctuation gauge fields which should appear in our ultimate effective theory. $\mathcal{L}_0$, however, collides with this symmetry of $\mathcal{H}'$ – if we replace $\mathbf{v}_{A(B)} \to \mathbf{v} \pm \mathbf{a}$ in (3.23), we realize that $\mathbf{v}$ and $\mathbf{a}$ are *coupled* through $\mathcal{L}_0$ in the general case $K_A \neq K_B$:

$$\begin{aligned} \mathcal{L}_0 \;\; &\to \;\; \frac{K_A + K_B}{4}\left((\nabla \times \mathbf{v})^2 + (\nabla \times \mathbf{a})^2\right) \\ &+ \;\; \frac{K_A - K_B}{2}(\nabla \times \mathbf{a}) \cdot (\nabla \times \mathbf{v}) \ . \end{aligned} \tag{3.24}$$

Therefore, via its coupling to $\mathbf{a}$, the superfluid velocity $\mathbf{v}$ couples to the "spin" of topological fermions and ultimately to the true *spin* of the real electrons. This is an unacceptable feature for the effective theory and seriously handicaps the general "$A - B$" gauge, in which the original phase is split into $\varphi \to \varphi_A + \varphi_B$, with $\varphi_{A(B)}$ each containing some *arbitrary* fraction of the original vortex defects. In contrast, the symmetrized transformation (3.4) which apportions vortex defects equally between φ_A and φ_B [58] leads to $K_A = K_B$ and to decoupling of $\mathbf{v}$ and $\mathbf{a}$ at quadratic order, thus eliminating the problem at its root. Furthermore, even if we start with the general

"$A-B$" gauge, the renormalization of $\mathcal{L}_0$ arising from integration over fermions will ultimately drive $K_A - K_B \to 0$ and make the coupling of $\mathbf{v}$ and $\mathbf{a}$ *irrelevant* in the RG sense [67]. This argument is actually quite rigorous in the case of quantum fluctuations where the symmetrized gauge (3.4) represents a fixed point in the RG analysis [58]. Consequently, it appears that the symmetrized singular transformation (3.4) employed in (3.10) is the *preferred* gauge for the construction of the effective low-energy theory. In this respect, while all the singular $A-B$ gauges are created equal some are ultimately more equal than others.

The above discussion provides the rationale behind using the FT transformation in our quest for the effective theory. At low energies, the interactions between quasiparticles and vortices are represented by two U(1) gauge fields v and a (3.20,3.21). The conversion of a from a Z_2-valued to a non-compact U(1) field with Maxwellian action is effected by the confinement of the Doppler to the Berry half of a singly quantized vortex – in the coarse-graining process the phase factors $\exp(i\delta)$ of the non-compact Doppler part "contaminate" the original (± 1) factors supplied by a (3.7). This contamination diminishes as doping $x \to 0$ since then $v_F/v_\Delta \to 0$ and "vortices" are effectively liberated of their Doppler content. In this limit, the pure Z_2 nature of a is recovered and one enters the realm of the Z_2 gauge theory of Senthil and Fisher [68]. In contrast, in the pairing pseudogap regime of this paper where $v_F/v_\Delta > 1$ and singly quantized ($hc/2e$) vortices appear to be the relevant excitations [69], we expect the effective theory to take the U(1) form described by v and a (3.20,3.21).

3.2.5 Maxwellian form of the Jacobian $\mathcal{L}_0[v_\mu, a_\mu]$

We shall now derive an expression for the long distance, low energy form of the "Jacobian" $\mathcal{L}_0[v_\mu, a_\mu]$ (see Eqn. 3.11) which serves as the "bare action" for the gauge fields v_μ and a_μ of our effective theory. As shown below, this form is a non-compact Maxwellian whose stiffness K (or inverse "charge" $1/e^2 = K$) stands in intimate relation to the helicity modulus tensor of a dSC and, in the pseudogap regime of strong superconducting fluctuations, can be expressed in terms of a *finite* physical superconducting correlation length $\xi_{\rm sc}$, $K \propto \xi_{\rm sc}^2$ (2D); $K \propto \xi_{\rm sc}$ ((2+1)D). Upon entering the superconducting phase and $\xi_{\rm sc} \to \infty$, $K \to \infty$ as well (or $e^2 \to 0$), implying that v_μ and a_μ have become massive. The straightforward relationship between the massless (or massive) character of $\mathcal{L}_0[v_\mu, a_\mu]$ and the superconducting phase disorder (or order) is a consequence of rather general physical and symmetry principles.

For the sake of simplicity, but without loss of generality, we consider first an example of an s-wave superconductor with a large gap Δ extending over all of the Fermi surface. The action takes the form similar to (3.10):

$$\tilde{\mathcal{L}} = \tilde{\Psi}^\dagger[(\partial_\tau + ia_\tau)\sigma_0 + iv_\tau\sigma_3]\tilde{\Psi} + \tilde{\Psi}^\dagger\mathcal{H}'_s\tilde{\Psi} + \mathcal{L}_0[v_\mu, a_\mu], \tag{3.25}$$

but with $\mathcal{H}'_s$ defined as:

$$\begin{pmatrix} \frac{1}{2m}(\hat{\pi} + \mathbf{v})^2 - \epsilon_F & \Delta \\ \Delta & -\frac{1}{2m}(\hat{\pi} - \mathbf{v})^2 + \epsilon_F \end{pmatrix} \quad , \tag{3.26}$$

with $\hat{\pi} = \hat{\mathbf{p}} + \mathbf{a}$. When the the electromagnetic field is included, v_μ in the fermionic part action is substituted by $v_\mu - (e/c)A_\mu$, while there is no substitution in $\mathcal{L}_0$.

Thermal vortex-antivortex fluctuations

In the language of BdG fermions the system (3.25) is a large gap "semiconductor" and the Berry gauge field couples to it minimally through "BdG" vector and scalar potentials $\mathbf{a}$ and a_τ. Such BdG semiconductor is a poor dielectric diamagnet with respect to a_μ. We proceed to ignore its "diamagnetic susceptibility" and also set $a_\tau = 0$ to concentrate on thermal fluctuations. The Berry gauge field part of the coupling between quasiparticles and vortices in a large gap s-wave superconductor influences the vortices only through weak short-range interactions which are unimportant in the region of strong vortex fluctuations near Kosterlitz-Thouless transition. We can therefore drop $\mathbf{a}$ from the fermionic part of the action and integrate over it to obtain $\mathcal{L}_0$ in terms of physical vorticity $\rho(\mathbf{r}) = (\nabla \times \mathbf{v})/\pi$. Additional integration over the fermions produces the effective free energy functional for vortices:

$$\mathcal{F}[\rho] = M^2(2\mathbf{v} - \frac{2e}{c}\mathbf{A}^{\rm ext})^2 + (\cdots) + \mathcal{L}_0[\rho] \quad , \tag{3.27}$$

where we have used our earlier notation and have introduced a small external transverse vector potential $\mathbf{A}^{\rm ext}$. The ellipsis denotes higher order contributions to vortex interactions. As discussed earlier in this Section, the familiar long range interactions between vortices lead directly to the standard Coulomb gas representation of the vortex-antivortex fluctuation problem and Kosterlitz-Thouless transition.

The presence of these long range interactions implies that the vortex system is incompressible (3.16) and long distance vortex density fluctuations are suppressed. When studying the coupling of BdG quasiparticles to these fluctuations it therefore suffices to expand the "entropic" part:

$$\mathcal{L}_0[\rho] \cong \frac{1}{2}\pi^2 K\delta\rho^2 + \dots = \frac{1}{2}K(\nabla\times\mathbf{v})^2 + \dots \quad . \tag{3.28}$$

The above expansion is justified above T_c since we know that at $T \gg T_c$ we must match the purely entropic form of a non-interacting particle system, $\mathcal{L}_0 \propto \delta\rho^2$ (see Appendix B.1).

To uncover the physical meaning of the coefficient K we expand the free energy F of the vortex system to second order in $\mathbf{A}^{\rm ext}$. In the pseudogap state the gauge invariance demands that F depends only on $\nabla\times\mathbf{A}^{\rm ext}$:

$$F[\nabla\times\mathbf{A}^{\rm ext}] = F[0] + \frac{(2e)^2}{2c^2}\chi\int d^2r(\nabla\times\mathbf{A}^{\rm ext})^2 + \dots \tag{3.29}$$

Note that $((2e)^2/c^2)\chi$ is just the diamagnetic susceptibility in the pseudogap state. χ determines the long wavelength form of the helicity modulus tensor $\Upsilon_{\mu\nu}(q)$ defined as:

$$\Upsilon_{\mu\nu}(q) = \Omega\frac{\delta^2 F}{\delta A^{\rm ext}_\mu(q)\delta A^{\rm ext}_\nu(-q)}\Big|_{A^{\rm ext}_\mu\to 0} \quad . \tag{3.30}$$

The above is the more general form of $\Upsilon_{\mu\nu}(q)$ applicable to uniaxially symmetric 3D and (2+1)D XY or Ginzburg-Landau models; in 2D only $\Upsilon_{xy}(\mathbf{q})$ appears. In the long wavelength limit $\Upsilon_{\mu\nu}(q)$ vanishes as $q^2\chi$:

$$\frac{c^2}{(2e)^2}\Upsilon_{\mu\nu}(\mathbf{q}) = \chi\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_\alpha q_\beta + \dots \quad , \tag{3.31}$$

for the isotropic case, while for the anisotropic situation $\chi_\perp \neq \chi_\parallel$:

$$\frac{c^2}{(2e)^2}\Upsilon_{\mu\nu}(\mathbf{q}) = (\chi_\parallel - \chi_\perp)\epsilon_{z\alpha\mu}\epsilon_{z\beta\nu}q_\alpha q_\beta + \chi_\perp\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_\alpha q_\beta \ . \tag{3.32}$$

$\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol, summation over repeated indices is understood and index z of the anisotropic 3D GL or XY model is replaced by τ for the (2+1)D case. $(2e)^2/c^2$ is factored out for later convenience.

Let us compute the helicity modulus of the problem explicitly. This is done by absorbing the small transverse vector potential $\mathbf{A}^{\rm ext}$ into $\mathbf{v}$ in (3.25,3.6) and integrating over the new variable $\mathbf{v} - (e/c)\mathbf{A}^{\rm ext}$. The hecility modulus tensor measures the screening properties of the vortex system. In a superconductor, with topological defects bound in vortex-antivortex dipoles, there is no screening at long distances. This translates into Meissner effect for $\mathbf{A}^{\rm ext}$. When the dipoles unbind and some free vortex-antivortex excitations appear the screening is now possible over all lengthscales and there is no Meissner effect for $\mathbf{A}^{\rm ext}$. The information on presence or absence of such screening is actually stored entirely in $\mathcal{L}_0$, where $\mathbf{A}^{\rm ext}$ re-emerges after the above change of variables. We finally obtain:

$$4\chi = K - \frac{K^2\pi^2}{T}\lim_{|\mathbf{q}|\to 0^+}\int d^2r e^{i\mathbf{q}\cdot\mathbf{r}}\langle\delta\rho(\mathbf{r})\delta\rho(0)\rangle \quad , \tag{3.33}$$

where the thermal average $\langle\cdots\rangle$ is over the free energy (3.27) with $\mathbf{A}^{\rm ext} = 0$ or, equivalently, (3.25). In the normal phase only the first term contributes in the long wavelength limit, the second being down by an extra power of q^2 courtesy of long range vortex interactions. Consequently, $K = 4\chi$ (the factor of 4 is due to the fact that the true superfluid velocity is $2\mathbf{v}$ [65]).

We see that in the pseudogap phase, with free vortex defects available to screen, $\mathcal{L}_0[\mathbf{v}]$ takes on a massless Maxwellian form, the stiffness of which is given by the diamagnetic susceptibility of a strongly fluctuating superconductor. In the fluctuation region χ is given by the superconducting correlation length $\xi_{\rm sc}$ [70]:

$$K = 4\chi = 4\mathcal{C}_2 T\xi_{\rm sc}^2 \quad , \tag{3.34}$$

where $\mathcal{C}_2$ is a numerical constant, intrinsic to a 2D GL, XY or some other model of superconducting fluctuations.

As we approach T_c, $\xi_{\rm sc}\to\infty$ and the stiffness of Maxwellian term (3.28) diverges. This can be interpreted as the Doppler gauge field becoming massive. Indeed, immediately bellow the Kosterlitz-Thouless transition at T_c, $\mathcal{L}_0 \cong m_v^2\mathbf{v}^2 + \ldots$, where

$m_v \ll M$ and $m_v \to 0$ as $T \to T_c^-$. This is just a reflection of the helicity modulus tensor now becoming *finite* in the long wavelength limit,

$$\Upsilon_{xy} \to \frac{4e^2}{c^2} \frac{m_v^2 M^2}{4M^2 + m_v^2}.$$

Topological defects are now bound in vortex-antivortex pairs and cannot screen resulting in the Meissner effect for $\mathbf{A}^{\rm ext}$. The system is a superconductor and $\mathbf{v}$ had become massive.

Returning to a d-wave superconductor, we can retrace the steps in the above analysis but we must replace $\mathcal{H}'_s$ in (3.25) with $\mathcal{H}'$ (3.6). Now, instead of a large gap BdG "semiconductor", we are dealing with a narrow gap "semiconductor" or BdG "semimetal" because of the low energy nodal quasiparticles. This means that we must restore the Berry gauge field $\mathbf{a}$ to the fermionic action since the contribution from nodal quasiparticles makes its BdG "diamagnetic susceptibility" very large, $\chi_{\rm BdG} \sim 1/T \gg 1/T^*$ (see the next Section). The long distance fluctuations of $\mathbf{v}$ and $\mathbf{a}$ are now *both* strongly suppressed, the former through incompressibility of the vortex system and the latter through $\chi_{\rm BdG}$. This allows us to expand (3.23):

$$\mathcal{L}_0 \cong \frac{K_A}{4}(\nabla \times \mathbf{v}_A)^2 + \frac{K_B}{4}(\nabla \times \mathbf{v}_B)^2 + (\cdots) \quad , \tag{3.35}$$

where $K_A = K_B = K$ is mandated by the FT singular gauge. Since in our gauge the fermion spin and charge channels decouple $\mathbf{A}^{\rm ext}$ still couples only to $\mathbf{v}$ and the above arguments connecting K to the helicity modulus and diamagnetic susceptibility χ follow through. This finally gives the Maxwellian form of Ref. [58]:

$$\mathcal{L}_0 \to \frac{K}{2}(\nabla \times \mathbf{v})^2 + \frac{K}{2}(\nabla \times \mathbf{a})^2 \quad , \tag{3.36}$$

where K is still given by Eq. (3.34). Note however that $\xi_{\rm sc}$ of a d-wave and of an s-wave superconductor are two rather *different* functions of T, x and other parameters of the problem, due to strong Berry gauge field renormalizations of vortex interactions in the d-wave case. Nonetheless, as $T \to T_c$, the Kosterlitz-Thouless critical behavior remains unaffected since $\chi_{\rm BdG}$, while large, is still finite at all finite T.

To summarize, we have shown that $\mathcal{L}_0[\rho_A, \rho_B] = \mathcal{L}_0[\mathbf{v}, \mathbf{a}]$ in the pseudogap state takes on the massless Maxwellian form (3.36), with the stiffness K set by the true superconducting correlation length $\xi_{\rm sc}$ (3.34). This result holds as a general feature of our theory irrespective of whether one employs a Ginzburg-Landau theory, XY model, vortex-antivortex Coulomb plasma, or any other description of strongly fluctuating dSC, as long as such description properly takes into account vortex-antivortex fluctuations and reproduces Kosterlitz-Thouless phenomenology. In the Appendix B.1 we show that within the continuum vortex-antivortex Coulomb plasma model:

$$\mathcal{L}_0 \to \frac{T}{2\pi^2 n_l}\left[(\nabla\times\mathbf{v})^2 + (\nabla\times\mathbf{a})^2\right] \quad , \tag{3.37}$$

where n_l is the average density of *free* vortex and antivortex defects. Comparison with (3.36) allows us to identify $\xi_{\rm sc}^{-2} \leftrightarrow 4\pi^2\mathcal{C}_2 n_l$ [58].

Quantum fluctuations of (2+1)D vortex loops

The above results can be generalized to quantum fluctuations of spacetime vortex loops. The superflow fields $v_{A(B)\mu}$ satisfy $(\partial\times v_{A(B)})_\mu = 2\pi j_{A(B)\mu}$, where $j_{A(B)\mu}(x)$ are the coarse-grained vorticities associated with $A(B)$ vortex defects and $\langle j_{A(B)\mu}\rangle = 0$. The topology of vortex loops dictates that $j_{A(B)\mu}(x)$ be a purely transverse field, i.e. $\partial\cdot j_{A(B)} = 0$, reflecting the fact that loops have no starting nor ending point. Again, we begin with a large gap s-wave superconductor and use its poor BdG diamagnetic/dielectric nature to justify dropping the Berry gauge field a_μ from the fermionic part of (3.25) and integrating over a_μ in the "entropic" part containing $\mathcal{L}_0[v_\mu, a_\mu]$.

The integration over the fermions contains an important novelty specific to the (2+1)D case: the appearance of the Berry phase terms for quantum vortices as they wind around fermions. Such Berry phase is the consequence of the first order time derivative in the original fermionic action (3.1). If we think of spacetime vortex loops as worldlines of some relativistic quantum bosons dual to the Cooper pair field $\Delta(\mathbf{r},\tau)$, as we do in the Appendix B.1, then these bosons see Cooper pairs and quasiparticles as sources of "magnetic" flux [71]. At the mean field level, this translates into a dual "Abrikosov lattice" or a Wigner crystal of holes in a dual superfluid. Accordingly, the non-superconducting ground state in the pseudogap regime will likely contain a weak charge modulation. We will ignore it for the rest of this paper. This is justified by the fact that a_μ does not couple to charge directly.

Hereafter, we assume that the transition from a dSC into a pseudogap phase proceeds via unbinding of vortex loops of a (2+1)D XY model or its GL counterpart or, equivalently, an anisotropic 3D XY or GL model where the role of imaginary time is taken on by a third spatial axis z [72]. We can now integrate out the fermions to obtain the effective Lagrangian for coarse-grained spacetime loops ($\pi j_\mu = (\partial \times v)_\mu$):

$$\mathcal{L}[j_\mu] = M_\mu^2 (2v_\mu - \frac{2e}{c} A_\mu^{\text{ext}})^2 + (\cdots) + \mathcal{L}_0[j_\mu] \quad , \tag{3.38}$$

where $M_x = M_y = M$ and $M_\tau = M/c_s$, with $c_s \sim v_F$ being the effective "speed of light" in the vortex loop spacetime. The incompressibility condition reads $\langle j_\mu(q) j_\nu(-q) \rangle \sim q^2[\delta_{\mu\nu} - (q_\mu q_\nu / q^2)]$ and in the pseudogap state permits the expansion:

$$\mathcal{L}_0[j_\mu] \cong \frac{1}{2}\pi^2 K_\tau j_\tau^2 + \frac{1}{2}\pi^2 \sum_{i=x,y} K_i j_i^2 \quad , \tag{3.39}$$

where $K_x = K_y = K \neq K_\tau$. Using the analogy with the uniaxially symmetric anisotropic 3D XY (or GL) model we can expand the ground state energy in the manner of (3.29):

$$E[\partial \times A^{\text{ext}}] = E[0] + \frac{(2e)^2}{2c^2} \sum_{\perp,\parallel} \chi_{\perp,\parallel} \int d^3x (\partial \times A^{\text{ext}})^2_{\perp,\parallel} \quad , \tag{3.40}$$

with $\chi_\perp = \chi_x = \chi_y = \chi$ and $\chi_\parallel = \chi_\tau \neq \chi$. The form of $\mathcal{L}_0$ (3.39) follows directly from the requirement that there are infinitely large vortex loops, resulting in vorticity fluctuations over all distances. χ and χ_τ determine the diamagnetic and dielectric susceptibilities of this insulating pseudogap state.

The explicit computation of $\Upsilon_{\mu\nu}(q)$ (3.30,3.32) leads to:

$$4\chi_{i,\tau} = K_{i,\tau} - K_{i,\tau}^2 \pi^2 \lim_{q \to 0^+} \int d^3x e^{iq\cdot x} \langle j(x)_{i,\tau} j(0)_{i,\tau} \rangle \quad , \tag{3.41}$$

where the second term is again eliminated by the incompressibility of the vortex system. This results in:

$$K = 4\chi = 4\mathcal{C}_3 \xi_\tau \quad ; \quad K_\tau = 4\chi_\tau = 4\mathcal{C}_3 \frac{\xi_{\text{sc}}^2}{\xi_\tau} \quad , \tag{3.42}$$

where we used the result for the anisotropic 3D XY or GL models: $\chi_\perp = \mathcal{C}_3 T \xi_\parallel$, $\chi_\parallel = \mathcal{C}_3 T \xi_\perp^2 / \xi_\parallel$, $\mathcal{C}_3$ being the intrinsic numerical constant for those models ($\mathcal{C}_3 \neq \mathcal{C}_2$).

In the case of (2+1)D vortex loops $\xi_\tau \propto \xi_{\rm sc}$ since our adopted model has the dynamical critical exponent $z = 1$.

The application to a d-wave superconductor is straightforward: the nodal structure of BdG quasiparticles in dSC helps along by the way of providing the anomalous stiffness for the Berry gauge field a_μ – upon integration over the nodal fermions the following term emerges in the effective action:

$$\frac{1}{2}\chi_{\rm BdG}(q)(q^2\delta_{\mu\nu} - q_\mu q_\nu)a_\mu(q)a_\nu(-q) \sim |q|[\delta_{\mu\nu} - (q_\mu q_\nu/q^2)]a_\mu(q)a_\nu(-q). \tag{3.43}$$

In the terminology of the BdG "semimetal", the "susceptibility" $\chi_{\rm BdG}$ is not merely very large, it diverges: $\chi_{\rm BdG} \sim 1/q$, as $q \to 0$. This allows us to expand $\mathcal{L}_0[j_{A\mu}, j_{B\mu}]$ and retain only quadratic terms:

$$\mathcal{L}_0 \cong \frac{K_{A\mu}}{4}(\partial \times v_A)^2_\mu + \frac{K_{B\mu}}{4}(\partial \times v_B)^2_\mu + (\cdots) \quad , \tag{3.44}$$

where $K_{A\mu} = K_{B\mu} = K_\mu$ is again assured by our choice of the FT singular gauge (3.4,3.11).

Finally, we rewrite (3.44) in terms of $v_\mu, a_\mu = (1/2)(v_{A\mu} \pm v_{B\mu})$:

$$\mathcal{L}_0 \to \frac{K_\mu}{2}(\partial \times v)^2_\mu + \frac{K_\mu}{2}(\partial \times a)^2_\mu \quad , \tag{3.45}$$

and observe that the fact that $A^{\rm ext}_\mu$ couples only to v_μ means that the expression (3.42) for K_μ is still valid. Of course, $\xi_{\rm sc}(x, T)$ is now truly different from its s-wave counterpart, including a possible difference in the critical exponent, since the coupling of d-wave quasiparticles to the Berry gauge field is marginal at the RG engineering level and may change the quantum critical behavior of the superconductor-pseudogap (insulator) transition.

3.3 Low energy effective theory

As indicated in Fig. 3.1, the low energy quasiparticles are located at the four nodal points of the d_{xy} gap function: $\mathbf{k}_{1,\bar{1}} = (\pm k_F, 0)$ and $\mathbf{k}_{2,\bar{2}} = (0, \pm k_F)$, hereafter denoted as $(1, \bar{1})$ and $(2, \bar{2})$ respectively. To focus on the leading low energy behavior

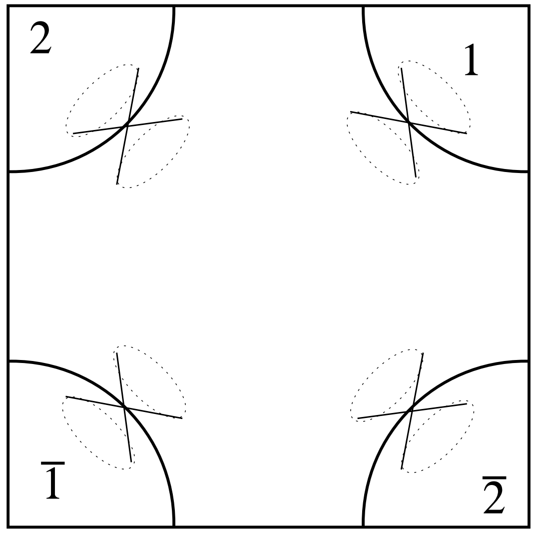


Figure 3.1: Schematic representation of the Fermi surface of the cuprate superconductors with the indicated nodal points of the $d_{x^2-y^2}$ gap.

of the fermionic excitations near the nodes we follow the standard procedure [73] and linearize the Lagrangian (3.10). To this end we write our TF spinor $\tilde{\Psi}$ as a sum of 4 nodal fermi fields,

$$\tilde{\Psi} = e^{i\mathbf{k}_1\cdot\mathbf{r}}\tilde{\Psi}_1 + e^{i\mathbf{k}_{\bar{1}}\cdot\mathbf{r}}\sigma_2\tilde{\Psi}_{\bar{1}} + e^{i\mathbf{k}_2\cdot\mathbf{r}}\tilde{\Psi}_2 + e^{i\mathbf{k}_{\bar{2}}\cdot\mathbf{r}}\sigma_2\tilde{\Psi}_{\bar{2}}. \tag{3.46}$$

The σ_2 matrices have been inserted here for convenience: they insure that we eventually recover the conventional form of the QED_3 Lagrangian. (Without the σ_2 matrices the Dirac velocities at $\bar{1}, \bar{2}$ nodes would have been negative.) Inserting $\tilde{\Psi}$ into (3.10) and systematically neglecting the higher order derivatives [73], we obtain the nodal Lagrangian of the form

$$\begin{aligned}\mathcal{L}_D &= \sum_{l=1,\bar{1}} \tilde{\Psi}_l^\dagger[D_\tau + iv_F D_x\sigma_3 + iv_\Delta D_y\sigma_1]\tilde{\Psi}_l \qquad (3.47)\\ &+ \sum_{l=2,\bar{2}} \tilde{\Psi}_l^\dagger[D_\tau + iv_F D_y\sigma_3 + iv_\Delta D_x\sigma_1]\tilde{\Psi}_l + \mathcal{L}_0[a_\mu],\end{aligned}$$

where $D_\mu = \partial_\mu + ia_\mu$ denotes the covariant derivative and $\mathcal{L}_0[a_\mu]$ is given by (3.45):

$$\mathcal{L}_0[a_\mu] = \frac{1}{2}K_\mu(\partial\times a)_\mu^2 \equiv \frac{1}{2e_\mu^2}(\partial\times a)_\mu^2 \quad . \tag{3.48}$$

$v_F = \partial\epsilon_{\mathbf{k}}/\partial\mathbf{k}$ denotes the Fermi velocity at the node and $v_\Delta = \partial\Delta_{\mathbf{k}}/\partial\mathbf{k}$ denotes the gap velocity. Note that v_F and v_Δ already contain renormalizations coming from high energy interactions and are effective material parameters of our theory. Similarly, $K_\mu = 1/e_\mu^2$, derived in the previous Section in terms of $\xi_{\rm sc}(x,T)$, are treated as adjustable parameters which are matched to experimentally available information on the range of superconducting correlations in the pseudogap state.

The massive Doppler gauge field v_μ does not appear in the above expression, since it merely generates short range interactions among the fermions.

In contrast, Berry gauge field a_μ remains *massless* in the pseudogap state, as it cannot acquire mass by coupling to the fermions. As seen from Eq. (3.47) a_μ couples minimally to the Dirac fermions and therefore its massless character is protected by gauge invariance. Physically, one can also argue that a_μ couples to the TF *spin* three-current – in a spin-singlet d-wave superconductor SU(2) spin symmetry must remain unbroken, thereby insuring that a_μ remains massless. Its massless Maxwellian dynamics (3.48) in the pseudogap state can therefore be traced back to the topological state of spacetime vortex loops and directly reflect the absence of true superconducting order or, equivalently, the presence of "vortex loop condensate" and dual order (Appendix B.1).

We have also dispensed with the residual interactions represented by $\mathcal{H}_{\rm res}$ in (3.2). These interactions are generically short-ranged contributions from the p-h and amplitude fluctuations part of the p-p channel and in our new notation are exemplified by:

$$\mathcal{H}_{\rm res} \to \frac{1}{2}\sum_{l,l'} I_{ll'}\tilde{\Psi}_l^\dagger\tilde{\Psi}_l\tilde{\Psi}_{l'}^\dagger\tilde{\Psi}_{l'} \quad . \tag{3.49}$$

The effective vertex $I_{ll'}$ has a scaling dimension -1 at the engineering level. This follows from the RG analysis near the massless Dirac points which sets the dimension of $\tilde{\Psi}_l$ to [length]$^{-1}$. The implication is that $I_{ll'}$ is *irrelevant* for low energy physics in the perturbative RG sense and we are therefore justified in setting $I_{ll'} \to 0$. However, the residual interactions are known to become important if stronger than some critical value I_c [74, 75]. In the present theory, this is bound to happen in the severely underdoped regime and at half-filling, as $x \to 0$ (see Section 3.6). In this case, the residual interactions are becoming large and comparable in scale to the pairing pseudogap Δ, and are likely to cause chiral symmetry breaking (CSB) which leads to spontaneous mass generation for massless Dirac fermions. The CSB and its variety of patterns in the context of the theory (3.47,3.48) in underdoped cuprates are discussed in Section 3.6. Here, we shall focus on the chirally symmetric phase and assume $I_{ll'} < I_c$, which we expect to be the case for moderate underdoping. Therefore, we can set $I_{ll'} \to 0$.

Finally, Eq. (3.47,3.48) represents the effective low energy theory for nodal TF. This theory describes the problem of massless topological fermions interacting with massless vortex "Beryons", i.e. the quanta of the Berry gauge field a_μ, and is formally equivalent to the Euclidean quantum electrodynamics of massless Dirac fermions in (2+1) dimensions (QED_3). It, however, suffers from an intrinsic Dirac anisotropy by virtue of $v_F \neq v_\Delta$.

3.3.1 Effective Lagrangian for the pseudogap state

Lagrangian (3.47) is not in the standard from as used in quantum electrodynamics where the matrices associated with the components of covariant derivatives form a Dirac algebra and mutually anticommute. In (3.47) the temporal derivative is associated with unit matrix and it therefore commutes (rather than anticommutes) with σ_1 and σ_3 matrices associated with the spatial derivatives. These nonstandard commutation relations, however, lead to some rather unwieldy algebra. For this reason we manipulate the Lagrangian (3.47) into a slightly different form that is consistent with usual field-theoretic notation. First, we combine each pair of antipodal (time reversed) two-component spinors into one 4-component spinor,

$$\Upsilon_1 = \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_{\bar{1}} \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} \tilde{\Psi}_2 \\ \tilde{\Psi}_{\bar{2}} \end{pmatrix}. \tag{3.50}$$

Second, we define a new adjoint four-component spinor

$$\bar{\Upsilon}_l = -i\Upsilon_l^\dagger \gamma_0. \tag{3.51}$$

In terms of this new spinor the Lagrangian becomes

$$\mathcal{L}_D = \sum_{l=1,2} \bar{\Upsilon}_l \gamma_\mu D_\mu^{(l)} \Upsilon_l + \frac{1}{2} K_\mu (\partial \times a)_\mu^2 \ , \tag{3.52}$$

with covariant derivatives

$$\begin{aligned} D_\mu^{(1)} &= i[(\partial_\tau + ia_\tau), v_F(\partial_x + ia_x), v_\Delta(\partial_y + ia_y)], \\ D_\mu^{(2)} &= i[(\partial_\tau + ia_\tau), v_F(\partial_y + ia_y), v_\Delta(\partial_x + ia_x)]. \end{aligned}$$

The 4×4 gamma matrices, defined as

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \ \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \end{aligned} \tag{3.53}$$

now form the usual Dirac algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{3.54}$$

and furthermore satisfy

$$\mathrm{Tr}(\gamma_\mu) = 0 \ , \quad \mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\delta_{\mu\nu}. \tag{3.55}$$

The use of the adjoint spinor $\bar{\Upsilon}$ instead of conventional $\Upsilon^\dagger$ is a purely formal device which will simplify calculations but does not alter the physical content of the theory. At the end of the calculation we have to remember to undo the transformation (3.51) by multiplying the $\langle \Upsilon(x)\bar{\Upsilon}(x')\rangle$ correlator by $i\gamma_0$ to obtain the physical correlator $\langle \Upsilon(x)\Upsilon^\dagger(x')\rangle$.

Next, to make the formalism simpler still we can eliminate the asymmetry between the two pairs of nodes by performing an internal SU(2) rotation at nodes 2, $\bar{2}$:

$$\Upsilon_2 \to e^{-i\frac{\pi}{4}\gamma_0} \gamma_1 \Upsilon_2, \tag{3.56}$$

leading to the anisotropic QED_3 Lagrangian

$$\mathcal{L}_D = \sum_{l=1,2} \bar{\Upsilon}_l v_\mu^{(l)} \gamma_\mu (i\partial_\mu - a_\mu) \Upsilon_l + \frac{1}{2} K_\mu (\partial \times a)_\mu^2 \ , \tag{3.57}$$

with $v_\mu^{(1)} = (1, v_F, v_\Delta)$ and $v_\mu^{(2)} = (1, v_\Delta, v_F)$.

We shall start by considering the isotropic case, $v_F = v_\Delta = 1$, which although unphysical in the strictest sense, is computationally much simpler and provides penetrating insights into the physics embodied by the QED_3 Lagrangian (3.57). After we have understood the isotropic case we will then be ready to tackle the calculation for the general case and will show that Dirac cone anisotropy does not modify the essential physics discussed here. To make contact with standard literature on QED_3, we further consider a more general problem with N *pairs* of nodes described by Lagrangian

$$\mathcal{L}_D = \sum_{l=1}^{N} \bar{\Upsilon}_l \gamma_\mu (i\partial_\mu - a_\mu) \Upsilon_l + \frac{1}{2} K_\mu (\partial \times a)_\mu^2 \quad . \tag{3.58}$$

For the basic problem of a single CuO_2 layer $N = 2$. As we will show in the next Section, N itself is variable and can be equal to four or six in bi- and multi-layer cuprates. Our analytic results can be viewed as arising from the formal $1/N$ expansion, although we expect them to be qualitatively (and even quantitatively!) accurate even for $N = 2$ as long as we are within the *symmetric* phase of QED_3 – the quantitative accuracy stems from a fortuitous conspiracy of small numerical prefactors [74].

3.3.2 Berryon propagator

Ultimately, we are interested in the properties of physical electrons. To describe those we need to understand the properties of the electron–electron interaction mediated by the gauge field a_μ. To this end we proceed to calculate the Berry gauge field propagator by integrating out the fermion degrees of freedom from the Lagrangian (3.58). To one loop order this corresponds to evaluating the vacuum polarization bubble Fig. 3.2(a). Employing the standard rules for Feynman diagrams in the momentum space [76] the vacuum polarization reads:

$$\Pi_{\mu\nu}(q) = N \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[G_0(k)\gamma_\mu G_0(k+q)\gamma_\nu]. \tag{3.59}$$

Here $G_0(k) = k_\alpha \gamma_\alpha / k^2$ is the free Dirac Green function, $k = (k_0, \mathbf{k})$ denotes the Euclidean three-momentum, and the trace is performed over the γ matrices.

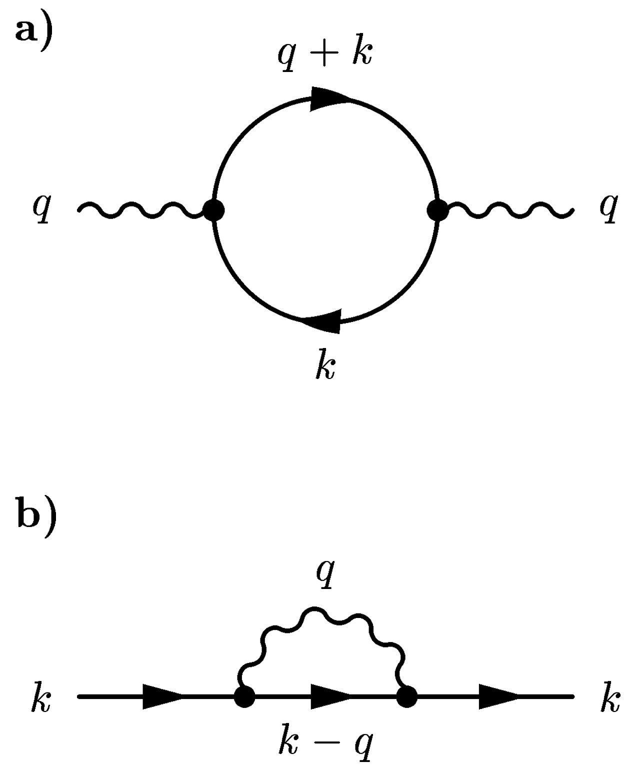


Figure 3.2: One loop Berryon polarization (a) and TF self energy (b).

The integral in Eq. (3.59) is a standard one (see Appendix B.2 for the details of computation) and the result is

$$\Pi_{\mu\nu}(q) = \frac{N}{8}|q|\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right), \tag{3.60}$$

where $|q| = \sqrt{q^2}$. The one loop effective action for the Berry gauge field therefore becomes $(2\pi)^{-3}\int d^3q\mathcal{L}_B$ with

$$\mathcal{L}_B[a] = \left(\frac{N}{8}|q| + \frac{1}{2e^2}q^2\right)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)a_\mu(q)a_\nu(-q)\ . \tag{3.61}$$

At low energies and long wavelengths, $|q|e^2 \ll N/4$, the fermion polarization completely overwhelms the original Maxwell bare action term and the Berryon properties become universal. In particular, the coupling constant $1/e^2$ drops out at low energies and only reappears as the ultraviolet cutoff. Physically, the medium of massless Dirac fermions screens the long range interactions mediated by a_μ. In QED_3 this screening is incomplete: the gauge field becomes stiffer by one power of q but still remains *massless*, in accordance with our general expectations. This anomalous stiffness of a_μ justifies the quadratic level expansion of $\mathcal{L}_0$ (3.45) and renders higher order terms *irrelevant* in the RG sense. The theory therefore clears an important self-consistency check.

At low energies the fully dressed Berryon propagator is given as an inverse of the polarization,

$$D_{\mu\nu}(q) = \Pi^{-1}_{\mu\nu}(q) \quad . \tag{3.62}$$

In order to perform this inversion we have to fix the gauge. To this end we implement the usual gauge fixing procedure by replacing $q_\mu q_\nu/q^2 \to (1-\xi^{-1})q_\mu q_\nu/q^2$ in Eq. (3.60). $\xi \geq 0$ parameterizes the orbit of all covariant gauges. For example, $\xi = 0$ corresponds to Lorentz gauge $k_\mu a_\mu(k) = 0$ while $\xi = 1$ corresponds to Feynman gauge. Upon inversion we obtain the low energy Berryon propagator

$$D_{\mu\nu}(q) = \frac{8}{|q|N}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1-\xi)\right), \tag{3.63}$$

in agreement with previous authors [81].

3.3.3 TF self energy and propagator

Topological fermion (TF) propagator is a gauge dependent entity and one could therefore immediately object that as such it has no direct physical content and is of no interest. While in the strictest sense, all observable quantities are gauge independent and the 2−point fermion function is manifestly *not* gauge independent, the dependence on the choice of gauge can be confined to the field strength renormalization [82]. In this Section we will show that in the leading order in large N expansion, the TF propagator is a powerlaw. The powerlaw exponent depends on the choice of gauge. Since we know that the theory flows to the strong coupling limit, we expect that the powerlaw exponent describing the physical spectrum is not trivial i.e. the poles in the spectrum are replaced by a branchcut. This means that the plane wave states acquire finite lifetime at arbitrarily low energy.

The lowest order ($O(1/N)$) self-energy diagram is depicted in Fig. 3.2(b) and reads

$$\Sigma(k) = \int \frac{d^3q}{(2\pi)^3} D_{\mu\nu}(q)\gamma_\mu G_0(k+q)\gamma_\nu. \tag{3.64}$$

Again, the computation is rather straightforward (see Appendix B.2 for details) and the most divergent part is

$$\Sigma(k) = \frac{4(2-3\xi)}{3\pi^2 N} \not{k} \ln\left(\frac{\Lambda}{|k|}\right), \tag{3.65}$$

where we have introduced the Feynman "slash" notation, $\not{k} = k_\mu \gamma_\mu$.

To the leading $1/N$ order, the inverse TF propagator is given by

$$G^{-1}(k) = \not{k} \left[1 + \eta \ln\left(\frac{\Lambda}{|k|}\right)\right] \tag{3.66}$$

with

$$\eta = -\frac{4(2-3\xi)}{3\pi^2 N} \quad . \tag{3.67}$$

Higher order contributions in $1/N$ will necessarily affect this result. Renormalization group arguments [78] and non-perturbative approaches [79] strongly suggest that Eq. (3.66) represents the start of a perturbative series that eventually resums into a power law:

$$G^{-1}(k) = \not{k} \left(\frac{\Lambda}{|k|}\right)^\eta \quad . \tag{3.68}$$

This implies real-space propagator of the form

$$G(r) = \Lambda^{-\eta} \frac{\not{r}}{r^{3+\eta}}. \tag{3.69}$$

Thus, the TF propagator exhibits a Luttinger-like algebraic singularity at small momenta, characterized by an anomalous exponent η. In the Lorentz gauge ($\xi = 0$) we find $\eta = -8/3\pi^2 N \simeq -0.13$, for $N = 2$. This rather small numerical value for the anomalous dimension exponent (which is even considerably smaller for $N = 4$ or $N = 6$) indicates that the unraveling of the Fermi liquid pole in the original TF propagator brought about by its interaction with the massless Berry gauge field is in a certain sense "weak". Note also that η is *negative* in the Lorentz gauge while it becomes positive, $\eta = 4/3\pi^2 N \simeq 0.06$ for $N = 2$, in the Feynman gauge ($\xi = 1$). The above results provide a strong indication that the physical, gauge-invariant fermion propagator also has a Luttinger-like form, characterized by a small and *positive* anomalous dimension [58].

3.4 Effects of Dirac anisotropy in symmetric QED_3

It is natural to examine to what extend is the theory modified by the inclusion of the Dirac anisotropy, i.e. the finite difference in the Fermi velocity v_F and the gap velocity v_Δ. In the actual materials the Dirac anisotropy $\alpha_D = \frac{v_F}{v_\Delta}$ decreases with decreasing doping from ~ 15 in the optimally doped to ~ 3 in the heavily underdoped samples.

There are two key issues: first, for a large enough number of Dirac fermion species N, how is the chirally symmetric infrared (IR) fixed point modified by the fact that $\alpha_D \neq 1$, and second, as we decrease N, does the chiral symmetry breaking occur at the same value of N as in the isotropic theory. In this Section we address in detail the first issue and defer the analysis of the second one to the future.

We determine the effect of the Dirac anisotropy, marginal by power counting, by the perturbative renormalization group (RG) to first order in the large N expansion. To the leading order δ in the small anisotropy $\alpha_D = 1 + \delta$, we obtain the analytic value of the RG β_{α_D}-function and find that it is proportional to δ, i.e. in the infra red the α_D decreases when $\delta > 0$ and the anisotropic theory flows to the isotropic fixed point. On the other hand, when $\delta < 0$, α_D increases in the IR and again the theory flows into the isotropic fixed point. These results hold even when anisotropy is not small as shown by numerical evaluation of the β-function. Upon integration of the RG equations we find small δ acquires an anomalous dimension $\eta_\delta = 32/5\pi^2 N$. Therefore, we conclude that the isotropic fixed point is stable against small anisotropy.

Furthermore, we show that in any covariant gauge renormalization of Σ due to the unphysical longitudinal degrees of freedom is *exactly* the same along any space-time direction. Therefore the only contribution to the RG flow of anisotropy comes from the physical degrees of freedom and our results for β_{α_D} stated above are in fact gauge invariant.

3.4.1 Anisotropic QED$_3$

In the realm of condensed matter physics there is no Lorentz symmetry to safeguard the space-time isotropy of the theory. Rather, the intrinsic Dirac anisotropy is always present since it ultimately arises from complicated microscopic interactions in the solid which eventually renormalize to band and pairing amplitude dispersion. Thus there is nothing to protect the difference in the Fermi velocity $v_F = \frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}}$ and the gap velocity $v_\Delta = \frac{\partial \Delta_{\mathbf{k}}}{\partial \mathbf{k}}$ from vanishing and, in fact, all HTS materials are anisotropic.

The value of α_D can be directly measured by the angle resolved photo-emission spectroscopy (ARPES), which is ultimately a "high" energy local probe of v_F and v_Δ. Since QED$_3$ is free on short distances, we can take the experimental values as the starting bare parameters of the field theory.

The pairing amplitude of the HTS cuprates has $d_{x^2-y^2}$ symmetry, and consequently there are four nodal points on the Fermi surface with Dirac dispersion around which we can linearize the theory. Note that the roles of x and y directions are interchanged between adjacent nodes. As before, we combine the four two-component Dirac spinors for the opposite (time reversed) nodes into two four-components spinors and label them as $(1,\bar{1})$ and $(2,\bar{2})$ (see Fig. 3.1).

Thus, the two-point vertex function of the non-interacting theory for, say, $1\bar{1}$ fermions is

$$\Gamma^{(2)free}_{1\bar{1}} = \gamma_0 k_0 + v_F \gamma_1 k_1 + v_\Delta \gamma_2 k_2. \tag{3.70}$$

Therefore the corresponding non-interacting "nodal" Green functions are

$$G_0^n(k) = \frac{\sqrt{g^n}_{\mu\nu}\gamma_\mu k_\nu}{k_\mu g_{\mu\nu} k_\nu} \equiv \frac{\gamma^n_\mu k_\mu}{k_\mu g_{\mu\nu} k_\nu}. \tag{3.71}$$

Here we introduced the (diagonal) "nodal" metric $g^{(n)}_{\mu\nu}$: $g^{(1)}_{00} = g^{(2)}_{00} = 1$, $g^{(1)}_{11} = g^{(2)}_{22} = v_F^2$, $g^{(1)}_{22} = g^{(2)}_{11} = v_\Delta^2$, as well as the "nodal" γ matrices γ^n. In what follows we assume that both v_F and v_Δ are dimensionless and that eventually one of them can be chosen to be unity by an appropriate choice of the "speed of light".

Since $\alpha_D \neq 1$ breaks Lorentz invariance of the theory and since it is the Lorentz invariance that protects the spacetime isotropy, we expect the β functions for α_D to acquire finite values. However, the theory still respects time-reversal and parity and for N large enough the system is in the chirally symmetric phase. These symmetries force the fermion self-energy of the interacting theory to have the form

$$\Sigma_{1\bar{1}} = A(k_{1\bar{1}}, k_{2\bar{2}}) \left(\gamma_0 k_0 + v_F \zeta_1 \gamma_1 k_1 + v_\Delta \zeta_2 \gamma_2 k_2 \right). \tag{3.72}$$

where $k_{1\bar{1}} \equiv k_\mu g^{(1)}_{\mu\nu} k_\nu$ and $k_{2\bar{2}} \equiv k_\mu g^{(2)}_{\mu\nu} k_\nu$. The coefficients ζ_i are in general different from unity. Furthermore, there is a discrete symmetry which relates flavors $1, \bar{1}$ and $2, \bar{2}$ and the x and y directions in such a way that

$$\Sigma_{2\bar{2}} = A(k_{2\bar{2}}, k_{1\bar{1}}) \left(\gamma_0 k_0 + v_\Delta \zeta_2 \gamma_1 k_1 + v_F \zeta_1 \gamma_2 k_2 \right). \tag{3.73}$$

In the computation of the fermion self-energy, this discrete symmetry allows us to concentrate on a particular pair of nodes without any loss of generality.

3.4.2 Gauge field propagator

As discussed above in the isotropic case, the effect of vortex-antivortex fluctuations at T=0 on the fermions can, at large distances, be included by coupling the nodal fermions minimally to a fluctuating $U(1)$ gauge field with a standard Maxwell action. Upon integrating out the fermions, the gauge field acquires a stiffness proportional to k, which is another way of saying that at the charged, chirally symmetric fixed point the gauge field has an anomalous dimension $\eta_A = 1$ (for discussion of this point in the bosonic QED see [80]).

We first proceed in the transverse gauge ($k_\mu a_\mu = 0$) which is in some sense the most physical one considering that the $\nabla \times \mathbf{a}$ is physically related to the vorticity, i.e. an intrinsically transverse quantity. We later extend our results to a general covariant gauge. To one-loop order the screening effects of the fermions on the gauge field are given by the polarization function

$$\Pi_{\mu\nu}(k) = \frac{N}{2} \sum_{n=1,2} \int \frac{d^3q}{(2\pi)^3} Tr[G^n_0(q) \gamma^n_\mu G^n_0(q+k) \gamma^n_\nu] \tag{3.74}$$

where the index n denotes the fermion "nodal" flavor. The above expression can be evaluated straightforwardly by noting that it reduces to the isotropic $\Pi_{\mu\nu}(k)$ once the integrals are properly rescaled [58]. The result can be conveniently presented by taking advantage of the "nodal" metric $g^n_{\mu\nu}$ as

$$\Pi_{\mu\nu}(k) = \sum_n \frac{N}{16 v_F v_\Delta} \sqrt{k_\alpha g^n_{\alpha\beta} k_\beta} \left(g^n_{\mu\nu} - \frac{g^n_{\mu\rho} k_\rho g^n_{\nu\lambda} k_\lambda}{k_\alpha g^n_{\alpha\beta} k_\beta} \right). \tag{3.75}$$

Note that this expression is explicitly transverse, $k_\mu \Pi_{\mu\nu}(k) = \Pi_{\mu\nu}(k) k_\nu = 0$, and symmetric in its space-time indices. It also properly reduces to the isotropic expression when $v_F = v_\Delta = 1$.

However, as opposed to the isotropic case, it is not quite as straightforward to determine the gauge field propagator $D_{\mu\nu}$. For example, as it stands the polarization matrix (3.75) is not invertible, which makes it necessary to introduce some gauge-fixing conditions. In our case the direct inversion of the 3×3 matrix would obscure the analysis and therefore, we choose to follow a more physical and notationally transparent line of reasoning which eventually leads to the correct expression for the gauge field propagator. Upon integrating out the fermions and expanding the effective action to the one-loop order, we find that

$$\mathcal{L}_{eff}[a_\mu] = (\Pi^{(0)}_{\mu\nu} + \Pi_{\mu\nu}) a_\mu a_\nu \tag{3.76}$$

where the bare gauge field stiffness is

$$\Pi^{(0)}_{\mu\nu} = \frac{1}{2e^2} k^2 \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \tag{3.77}$$

At this point we introduce the dual field b_μ which is related to a_μ as

$$b_\mu = \epsilon_{\mu\nu\lambda} q_\nu a_\lambda. \tag{3.78}$$

Physically, the b_μ field represents vorticity and we integrate over all possible vorticity configurations with the restriction that b_μ is transverse. We note that

$$\mathcal{L}[b_\mu] = \chi_0 b_0^2 + \chi_1 b_1^2 + \chi_2 b_2^2 \tag{3.79}$$

where χ_μ's are functions of k_μ and upon a straightforward calculation they can be found to read

$$\chi_\mu = \frac{1}{2e^2} + \frac{N}{16 v_F v_\Delta} \sum_{n=1,2} \frac{g^n_{\nu\nu} g^n_{\lambda\lambda}}{\sqrt{k_\alpha g^n_{\alpha\beta} k_\beta}} \quad , \tag{3.80}$$

where $\mu \neq \nu \neq \lambda \in \{0,1,2\}$. At low energies we can neglect the non-divergent bare stiffness and thus set $1/e^2 = 0$ in the above expression.

The expression (3.79) is manifestly gauge invariant and has the merit of not only being quadratic but also diagonal in the individual components of b_μ. Thus, integration over the vorticity (even with the restriction of transverse b_μ) is simple and we can easily determine the b_μ field correlation function

$$\langle b_\mu b_\nu \rangle = \frac{\delta_{\mu\nu}}{\chi_\mu} - \frac{k_\mu k_\nu}{\chi_\mu \chi_\nu} \left(\sum_i \frac{k_i^2}{\chi_i} \right)^{-1} . \tag{3.81}$$

The repeated indices are not summed in the above expression. Note that, in addition to being transverse, $\langle b_\mu b_\nu \rangle$ is also symmetric in its space time indices.

It is now quite simple to determine the correlation function for the a_μ field and in the transverse gauge we obtain

$$D_{\mu\nu}(q) = \langle a_\mu a_\nu \rangle = \epsilon_{\mu i j} \epsilon_{\nu k l} \frac{q_i q_k}{q^4} \langle b_j b_l \rangle. \tag{3.82}$$

Using the transverse character of $\langle b_\mu b_\nu \rangle$ (which is independent of the gauge) the above expression can be further reduced to

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left(\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \langle b^2 \rangle - \langle b_\mu b_\nu \rangle \right) . \tag{3.83}$$

It can be easily checked that in the isotropic limit the expression (3.83) properly reduces to the results obtained in a different way.

We can further extend this result to include a general gauge by writing

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left(\left(\delta_{\mu\nu} - (1 - \frac{\xi}{2}) \frac{q_\mu q_\nu}{q^2} \right) \langle b^2 \rangle - \langle b_\mu b_\nu \rangle \right) . \tag{3.84}$$

where ξ is our continuous parameterization of the gauge fixing. This expression can be justified by the Fadeev-Popov type of procedure starting from the Lagrangian

$$\mathcal{L} = \left(\Pi_{\mu\nu} + \frac{1}{\xi} \frac{2k^2}{\langle b^2 \rangle} \frac{k_\mu k_\nu}{k^2} \right) a_\mu a_\nu, \tag{3.85}$$

where the stiffness for the unphysical modes was judiciously chosen to scale as k in a particular combination of the physical scalars of the theory. Note that $\langle b^2 \rangle$ can be determined without ever considering gauge fixing terms. In this way, the extension of a transverse gauge $\xi = 0$ to a general covariant gauge is accomplished by a simple substitution $k_\mu k_\nu \rightarrow (1 - \frac{\xi}{2}) k_\mu k_\nu$. The expression (3.84) is our final result for the gauge field propagator in a covariant gauge.

3.4.3 TF self energy

As discussed above, Σ is not gauge invariant in that it has an explicit dependence on the gauge fixing parameter. As we will show in this Section (and more generally in the Appendix B.3), the renormalization of Σ by the unphysical longitudinal degrees of freedom does not depend on the space-time direction: the term in Σ which is proportional to γ_0 is renormalized the same way by the gauge dependent part of the action as the terms proportional to γ_1 and γ_2. Therefore, the only contribution to the RG flow of α_D comes from the physical degrees of freedom.

We denote the topological fermion self-energy at the node n by $\Sigma_n(q)$. Hence, to the leading order in large N expansion we have

$$\Sigma_n(q) = \int \frac{d^3k}{(2\pi)^3} \gamma_\mu^n G_0^n(q-k) \gamma_\nu^n D_{\mu\nu}(k). \tag{3.86}$$

or explicitly

$$\Sigma_n(q) = \int \frac{d^3k}{(2\pi)^3} \gamma_\mu^n \frac{(q-k)_\lambda \gamma_\lambda^n}{(q-k)_\mu g_{\mu\nu} (q-k)_\nu} \gamma_\nu^n D_{\mu\nu}(k), \tag{3.87}$$

where the gauge field propagator $D_{\mu\nu}$ is already screened by the nodal fermions (3.84). Using the fact that

$$\gamma_\mu \gamma_\lambda \gamma_\nu = i\epsilon_{\mu\lambda\nu} \gamma_5 \gamma_3 + \delta_{\mu\lambda} \gamma_\nu - \delta_{\mu\nu} \gamma_\lambda + \delta_{\lambda\nu} \gamma_\mu, \tag{3.88}$$

where $\mu, \nu, \lambda \in \{0, 1, 2\}$ and $\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3$, we can easily see that

$$\gamma_\mu^n \gamma_\lambda^n \gamma_\nu^n D_{\mu\nu} = (2g_{\lambda\mu}^n \gamma_\nu^n - \gamma_\lambda^n g_{\mu\nu}^n) D_{\mu\nu}, \tag{3.89}$$

where we used the symmetry of the gauge field propagator tensor $D_{\mu\nu}$. Thus,

$$\Sigma_n(q) = \int \frac{d^3k}{(2\pi)^3} \frac{(q-k)_\lambda (2g_{\lambda\mu}^n \gamma_\nu^n - \gamma_\lambda^n g_{\mu\nu}^n) D_{\mu\nu}(k)}{(q-k)_\mu g_{\mu\nu}^n (q-k)_\nu} \tag{3.90}$$

and as shown in the Appendix B.3 at low energies this can be written as

$$\Sigma_n(q) = -\sum_\mu \eta^n_\mu(\gamma^n_\mu q_\mu)\ln\left(\frac{\Lambda}{\sqrt{q_\alpha g^n_{\alpha\beta}q_\beta}}\right). \tag{3.91}$$

Here Λ is an upper cutoff and the coefficients η are functions of the bare anisotropy, which have been reduced to a quadrature (see Appendix B.3). It is straightforward, even if somewhat tedious, to show that in case of weak anisotropy ($v_F = 1+\delta,\ v_\Delta = 1$), to order δ^2,

$$\eta^{1\bar{1}}_0 = -\frac{8}{3\pi^2 N}\left(1-\frac{3}{2}\xi-\frac{1}{35}\left(40-7\xi\right)\delta^2\right) \tag{3.92}$$

$$\eta^{1\bar{1}}_1 = -\frac{8}{3\pi^2 N}\left(1-\frac{3}{2}\xi+\frac{6}{5}\delta-\frac{1}{35}\left(43-7\xi\right)\delta^2\right) \tag{3.93}$$

$$\eta^{1\bar{1}}_2 = -\frac{8}{3\pi^2 N}\left(1-\frac{3}{2}\xi-\frac{6}{5}\delta-\frac{1}{35}\left(1-7\xi\right)\delta^2\right). \tag{3.94}$$

In the isotropic limit ($v_F = v_\Delta = 1$) we regain $\eta^n_\mu = -8(1-\frac{3}{2}\xi)/3\pi^2 N$ as previously found by others.

3.4.4 Dirac anisotropy and its β function

Before plunging into any formal analysis, we wish to discuss some immediate observations regarding the RG flow of the anisotropy. Examining the Eq. (3.91) it is clear that if $\eta^n_1 = \eta^n_2$ then the anisotropy *does not* flow and remains equal to its bare value. That would mean that anisotropy is marginal and the theory flows into the anisotropic fixed point. In fact, such a theory would have a *critical line* of α_D. For this to happen, however, there would have to be a symmetry which would protect the equality $\eta^n_1 = \eta^n_2$. For example, in the isotropic QED$_3$ the symmetry which protects the equality of η's is the Lorentz invariance. In the case at hand, this symmetry is broken and therefore we expect that η^n_1 will be different from η^n_2, suggesting that the anisotropy flows away from its bare value. If we start with $\alpha_D > 1$ and find that $\eta^{1\bar{1}}_2 > \eta^{1\bar{1}}_1$ at some scale $p < \Lambda$, we would conclude that the anisotropy is marginally irrelevant and decreases towards 1. On the other hand if $\eta^{1\bar{1}}_2 < \eta^{1\bar{1}}_1$, then anisotropy continues increasing beyond its bare value and the theory flows into a *critical point* with (in)finite anisotropy.

The issue is further complicated by the fact that η^n_μ is not a gauge invariant quantity, i.e. it depends on the gauge fixing parameter ξ. The statement that, say $\eta^n_1 > \eta^n_2$, makes sense only if the ξ dependence of η^n_1 and η^n_2 is exactly the same, otherwise we could choose a gauge in which the difference $\eta^n_2 - \eta^n_1$ can have either sign. However, we see from the equations (3.92-3.94) that in fact the ξ dependence of all η's is indeed the same. Although it was explicitly demonstrated only to the $\mathcal{O}(\delta^2)$, in the Appendix B.3 we show that it is in fact true to all orders of anisotropy for any choice of covariant gauge fixing. This fact provides the justification for our procedure. Now we supply the formal analysis reflecting the above discussion.

The renormalized 2-point vertex function is related to the "bare" 2-point vertex function via a fermion field rescaling Z_ψ as

$$\Gamma^{(2)}_R = Z_\psi \Gamma^{(2)}. \tag{3.95}$$

It is natural to demand that for example at nodes 1 and $\bar{1}$ at some renormalization scale p, $\Gamma^{(2)}_R(p)$ will have the form

$$\Gamma^{(2)}_R(p) = \gamma_0 p_0 + v^R_F \gamma_1 p_1 + v^R_\Delta \gamma_2 p_2. \tag{3.96}$$

Thus, the equation (3.96) corresponds to our renormalization condition through which we can eliminate the cutoff dependence and calculate the RG flows.

To the order of $1/N$ we can write

$$\Gamma^{(2)}_R(p) = Z_\psi \gamma^n_\mu p_\mu \left(1 + \eta^n_\mu \ln \frac{\Lambda}{p}\right) \tag{3.97}$$

where we used the fermionic self-energy (3.91). Multiplying both sides by γ_0 and tracing the resulting expression determines the field strength renormalization

$$Z_\psi = \frac{1}{1 + \eta^n_0 \ln \frac{\Lambda}{p}} \approx 1 - \eta^n_0 \ln \frac{\Lambda}{p}. \tag{3.98}$$

We can now determine the renormalized Fermi and gap velocities

$$\frac{v^R_F}{v_F} \approx (1 - \eta^{1\bar{1}}_0 \ln \frac{\Lambda}{p})(1 + \eta^{1\bar{1}}_1 \ln \frac{\Lambda}{p}) \approx 1 - (\eta^{1\bar{1}}_0 - \eta^{1\bar{1}}_1) \ln \frac{\Lambda}{p} \tag{3.99}$$

and

$$\frac{v^R_\Delta}{v_\Delta} \approx (1 - \eta^{1\bar{1}}_0 \ln \frac{\Lambda}{p})(1 + \eta^{1\bar{1}}_2 \ln \frac{\Lambda}{p}) \approx 1 - (\eta^{1\bar{1}}_0 - \eta^{1\bar{1}}_2) \ln \frac{\Lambda}{p}. \tag{3.100}$$

The corresponding renormalized Dirac anisotropy is therefore

$$\alpha_D^R \equiv \frac{v_F^R}{v_\Delta^R} \approx \alpha_D(1 - (\eta_2^{1\bar{1}} - \eta_1^{1\bar{1}}) \ln \frac{\Lambda}{p}). \tag{3.101}$$

The RG beta function can now be determined

$$\beta_{\alpha_D} = \frac{d\alpha_D^R}{d \ln p} = \alpha_D(\eta_2^{1\bar{1}} - \eta_1^{1\bar{1}}). \tag{3.102}$$

In the case of weak anisotropy ($v_F = 1 + \delta,\ v_\Delta = 1$) the above expression can be determined analytically as an expansion in δ. Using Eqs.(3.93-3.94) we obtain

$$\beta_{\alpha_D} = \frac{8}{3\pi^2 N}\left(\frac{6}{5}\delta(1+\delta)(2-\delta) + \mathcal{O}(\delta^3)\right). \tag{3.103}$$

Note that this expression is independent of the gauge fixing parameter ξ. For $0 < \delta \ll 1$ the β function is positive which means that anisotropy decreases in the IR and thus the anisotropic QED_3 scales to an isotropic QED_3. For $-1 \ll \delta < 0$ the β function is negative and in this case the anisotropy increases towards the fixed point $\alpha_D = 1$, i.e. again towards the isotropic QED_3. Note that for $\delta > 2$, $\beta < 0$ which may naively indicate that there is a fixed point at $\delta = 2$; this however cannot be trusted as it is outside of the range of validity of the power expansion of η_μ. The numerical evaluation of the β-function shows that, apart from the isotropic fixed point and the unstable fixed point at $\alpha_D = 0$, β_{α_D} does not vanish (see Fig. 3.3). This indicates that to the leading order in the $1/N$ expansion, the theory flows into the isotropic fixed point where $\alpha_D - 1$ has a scaling dimension $\eta_\delta = 32/(5\pi^2 N) > 1$.

3.5 Finite temperature extensions of QED_3

In this Section we focus on thermodynamics and spin response of the pseudogap state. First, in order to derive thermodynamics and spin susceptibility from the QED_3 theory [58] we generalize its form to finite T. This is a matter of some subtlety since, the moment $T \neq 0$, the theory loses its fictitious "relativistic invariance". Second, we show that the theory predicts a finite T scaling form for thermodynamic quantities in the pseudogap state. Third, we explicitly compute the leading $T \neq 0$

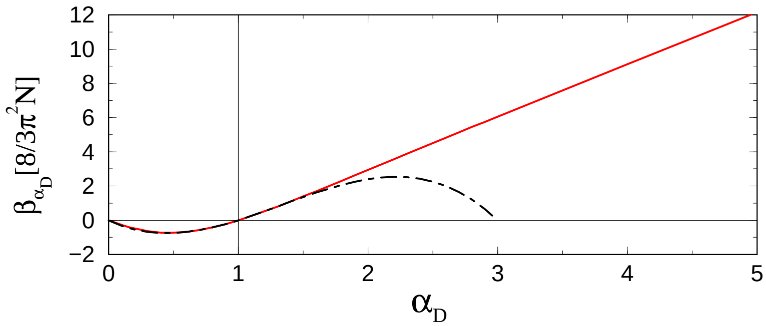


Figure 3.3: The RG β-function for the Dirac anisotropy in units of $8/3\pi^2 N$. The solid line is the numerical integration of the quadrature in the Eq. (B.52) while the dash-dotted line is the analytical expansion around the small anisotropy (see Eq. (3.92-3.94)). At $\alpha_D = 1$, β_{α_D} crosses zero with positive slope, and therefore at large length-scales the anisotropic QED_3 scales to an isotropic theory.

scaling and demonstrate that the deviations from "relativistic invariance" are actually *irrelevant* for T much less than the pseudogap temperature T^*, in the sense that the leading order $T \neq 0$ scaling of thermodynamic functions remains that of the *finite-T symmetric* QED_3. These deviations from "relativistic invariance" do, however, affect higher order terms. We illustrate these general results by computing the leading ($\sim T^2$) and next-to-the leading ($\sim T^3$) terms in specific heat c_v. Finally, we evaluate magnetic spin susceptibility χ_u and show that it is bounded by T^2 at low T but crosses over to $\sim T$ at higher T, closer to T^*. Consequently, the Wilson ratio $\chi_u T/c_v$ vanishes as $T \to 0$ implying the non-Fermi liquid nature of the pseudogap state in cuprates.

We start by noting that the spin susceptibility of a BCS-like d-wave superconductor vanishes linearly with temperature. The way to understand this result is to notice that the spin part of the ground state wavefunction, being a spin singlet, remains unperturbed by the application of a weak *uniform* magnetic field. However, the excited quasiparticle states are *not* in general spin singlets and therefore contribute to the thermal average of the spin susceptibility. Because their density of states is linear at low energies, at finite temperature the number of quasiparticles that are excited is $\sim k_B T$, each contributing a constant to the Pauli-like uniform spin susceptibility χ_u. Thus $\chi_u \sim T$.

When the superconducting order is destroyed by proliferation of unbound quantum $hc/2e$ vortices, the low-energy quasiparticle excitations are strongly interacting. The interaction originates from the fact that it is the spin singlet *pairs* that acquire a *one unit* of angular momentum in their center of the mass coordinate, carried by a $hc/2e$ vortex. This translates into a topological frustration in the propagation of the "spinon" excitations. As a result, non-trivial spin correlations persist in the excited states of the phase-disordered d-wave superconductor. At low temperature, this leads to a suppression of χ_u relative to that of the non-interacting quasiparticles. We shall argue below that in the phase-disordered superconductor, $\chi_u \sim T^2$ at low temperatures.

Similarly, in a d-wave superconductor, linear density of the quasiparticle states translates into T^2 dependence of the low T specific heat. When the interactions between quasiparticles are included the spectral weight is transferred to multi-particle states. Within QED$_3$ theory, however, the strongly interacting IR (infra-red) fixed point possesses emergent "relavistivistic invariance" at long distances and low energies and the dynamical critical exponent $z = 1$. Furthermore, the effective quantum action for vortices, deep in the phase disordered pseudogap state, introduces an additional lengthscale, the superconducting correlation length $\xi_{\tau,\perp}$ (labels τ and $\perp$ stand for time- and space-like, respectively). At $T = 0$ this scale serves as a short distance cutoff of the theory and possesses some degree of doping (x) dependence within the pseudogap state. We then argue that under rather general circumstances the low T electronic specific heat scales as T^2 while the free energy goes as T^3.

Deep in the phase disordered pseudogap state the fluctuations in the vorticity 3-vector $b_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$ are described by the "bare" Lagrangian of the QED$_3$ theory [58]:

$$\mathcal{L}_0(a_\mu) = \frac{K_\perp}{2} f_\perp\left(\frac{T}{k}, \frac{T}{\omega}, TK_{\perp,\tau}\right) b_0^2 + \frac{K_\tau}{2} f_\tau\left(\frac{T}{k}, \frac{T}{\omega}, TK_{\perp,\tau}\right) \vec{b}^2, \qquad (3.104)$$

where $f_\tau(0,0,0) = f_\perp(0,0,0) = 1$. $f_{\tau,\perp}$ are general scaling functions describing how $\mathcal{L}_0(a_\mu)$ is modified from its "relativistically invariant" $T = 0$ form as the temperature is turned on, and $K_{\perp,\tau}$ is related to the superconducting correlation length as $K_\tau \propto \xi_{sc}^2/\xi_\tau$ and $K_\perp \propto \xi_\tau$ [58]. Physically, such modifications are due to changes in the pattern of vortex-antivortex fluctuations induced by finite T.

To handle the intrinsic space-time anisotropy, it is convenient to introduce two tensors

$$
\begin{aligned}
A_{\mu\nu} &= \left(\delta_{\mu 0} - \frac{k_\mu k_0}{k^2}\right) \frac{k^2}{\vec{k}^2} \left(\delta_{0\nu} - \frac{k_0 k_\nu}{k^2}\right) \ , \\
B_{\mu\nu} &= \delta_{\mu i} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \delta_{j\nu} \ ,
\end{aligned} \tag{3.105}
$$

and rewrite the gauge field action as

$$
\mathcal{L}_0(a_\mu) = \frac{1}{2}\Pi^0_A a_\mu A_{\mu\nu} a_\nu + \frac{1}{2}\Pi^0_B a_\mu B_{\mu\nu} a_\nu \ . \tag{3.106}
$$

For details see Appendix B.4. It is straightforward to show that

$$
\begin{aligned}
\Pi^0_A &= K_\tau f_\tau \left(\frac{T}{k}, \frac{T}{\omega}, TK_{\perp,\tau}\right) (\vec{k}^2 + \omega^2), \\
\Pi^0_B &= K_\tau f_\tau \left(\frac{T}{k}, \frac{T}{\omega}, TK_{\perp,\tau}\right) \omega^2 + K_\perp f_\perp \left(\frac{T}{k}, \frac{T}{\omega}, TK_{\perp,\tau}\right) \vec{k}^2.
\end{aligned} \tag{3.107}
$$

The gauge field a_μ couples minimally to the Dirac fermions representing nodal BCS quasiparticles [58]. Consequently, the resulting Lagrangian reads

$$
\mathcal{L} = \bar{\psi}\left(i\gamma_\mu \partial_\mu + \gamma_\mu a_\mu\right)\psi + \mathcal{L}_0(a_\mu) \ . \tag{3.108}
$$

The integration over Berry gauge field a_μ reproduces the interaction among quasiparticles arising from the topological frustration referred to earlier.

3.5.1 Specific heat and scaling of thermodynamics

The only lengthscales that appear in the thermodynamics are the thermal length $\sim 1/T$, and the superconducting correlation lengths $K_{\perp,\tau}$. At $T = 0$, the two correlation lengths enter only as short distance cutoffs for the theory since the electronic action is controlled by the IR fixed point of QED$_3$. These observations allow us to write down the general scaling form for the free energy

$$
\mathcal{F}(T; x) = T^3 \Phi\big(K_\tau(x)T, K_\perp(x)T\big) \ . \tag{3.109}
$$

In the above scaling form $K_\perp$ is related to the $T \to 0$ *finite* superconducting correlation length of the pseudogap state $\xi_{\rm sc}(x)$. The ratio $K_\tau/K_{\rm sc}$ describes the anisotropy between time-like and space-like vortex fluctuations and is also a function of doping x. The scaling expressions for other thermodynamic functions can be derived from (3.109) by taking appropriate temperature derivatives. For details see Appendices B.5 and B.6.

We are interested in the limit of the thermal length $1/T$ being much longer than K_τ and $K_\perp$ i.e. in the limit of $\Phi(x \to 0, y \to 0)$. This is precisely the limit in which the free energy approaches the free energy of the finite temperature QED$_3$. Therefore, $\Phi(x, y)$ is regular at $x = y = 0$ (see Ref.[81]) and so in the limit of $T \to 0$, $\mathcal{F} \sim T^3$ or $c_v \sim T^2$.

3.5.2 Uniform spin susceptibility

The fermion fields ψ were defined in Ref. [58] where it is shown that the physical spin density $\psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow$ is equal to the Dirac fermion density $\bar{\psi}\gamma_0\psi$.

To compute the spin-spin correlation function $\langle S_z(-k) S_z(k) \rangle$ we introduce an auxiliary source $J_\mu(k)$ and couple it to fermion three current. Thus

$$\mathcal{L}[\bar{\psi}, \psi, a_\mu, J_\mu] = \bar{\psi}\left(i\gamma_\mu\partial_\mu + \gamma_\mu(a_\mu + J_\mu)\right)\psi + \mathcal{L}_0(a_\mu) \tag{3.110}$$

and since it is the z-component of the spin that couples to the gauge field we have

$$\langle S_z(-k) S_z(k) \rangle = \frac{1}{Z[J_\mu]} \frac{\delta}{\delta J_0(-k)} \frac{\delta}{\delta J_0(k)} Z[J_\mu] \Big|_{J_\mu = 0}, \tag{3.111}$$

Z being the quantum partition function. Now we let $a'_\mu = a_\mu + J_\mu$ and integrate out both the fermions and the gauge field a'_μ. The correlations between a'_μ fields are described by the polarization matrix which to the order of $1/N$ can be written in the form $\Pi_{\mu\nu} = (\Pi_A^0 + \Pi_A^F) A_{\mu\nu} + (\Pi_B^0 + \Pi_B^F) B_{\mu\nu}$. The resulting spin correlation function is then readily found to be

$$\langle S_z(-k) S_z(k) \rangle = \frac{\Pi_A^F \Pi_A^0}{\Pi_A^F + \Pi_A^0} \frac{\vec{k}^2}{\vec{k}^2 + \omega^2}, \tag{3.112}$$

where Π_A^F denotes the fermion current polarization function. Due to the scale invariance of the (massless) fermion action, the time component of the retarded polarization function has the scaling form

$$\Pi_A^{F\,ret}(q, \omega, T) = NT\, \mathcal{P}_A^F\left(\frac{q}{T}, \frac{\omega}{T}\right), \tag{3.113}$$

where $\mathcal{P}_A^F(x, y)$ is a universal function of its arguments and N is the number of the 4−component fermion species (see Appendix B.5). In the static limit, $\omega \to 0$,

$$\lim_{y\to 0} \Re e\, \mathcal{P}_A^F(x, y) = \frac{2\ln 2}{\pi} + \frac{x^2}{24\pi} + \mathcal{O}(x^3); \quad x \ll 1 \tag{3.114}$$

and

$$\lim_{y\to 0} \Re e\, \mathcal{P}_A^F(x, y) = \frac{x}{8} + \frac{6\zeta(3)}{\pi x^2} + \mathcal{O}(x^{-3}); \quad x \gg 1 \tag{3.115}$$

while

$$\lim_{y\to 0} \Im m\, \mathcal{P}_A^F(x, y) = \frac{2\ln 2}{\pi}\frac{y}{x} + \mathcal{O}\left(\frac{y^2}{x^2}\right); \quad x \ll 1 \tag{3.116}$$

In addition, in the limit of $\omega = 0$ and $k \to 0$ we have

$$f_\tau\left(\frac{T}{k}, \infty, K_\tau T\right) \to 1 + c\frac{T^2}{k^2} \tag{3.117}$$

where c is a pure number. Combining all of these results together we have

$$\chi(\omega = 0, q \to 0) = \frac{2N\ln 2}{\pi}\frac{K_\tau T^2}{\frac{2N\ln 2}{\pi c} + K_\tau T}. \tag{3.118}$$

There are several sources of correction to the scaling, and it is impossible to address all of them without an explicit model for the vortices. Instead, we will concentrate on the particular case of the corrections to scaling due to the Dirac cone anisotropy. It was shown in the previous Section that the Dirac cone anisotropy $\alpha_D = v_F/v_\Delta$ scales to unity at the QED$_3$ IR fixed point. Alternatively, when defined as $\alpha_D \equiv 1 + \delta$, δ has a scaling dimension $\eta_\delta > 1$. To the leading order in $1/N$, $\eta_\delta = 32/(5\pi^2 N)$ (see the previous Section). For simplicity, we assume that the bare action for the gauge field is $e^{-2}F_{\mu\nu}^2$. Then the T dependence of the free energy scales as

$$F = \frac{T^3}{v_F^2}\Phi\left(\frac{T}{e^2}, \frac{v_F}{v_\Delta}\right) = \frac{T^3}{v_F^2}\Phi\left(\frac{T}{e^2}, 1 + \delta(T)\right) \tag{3.119}$$

Thus, in the limit of $T \to 0$ we have

$$F = \frac{T^3}{v_F^2}\Phi\left(0,1\right) + \frac{T^{3+\eta_\delta}}{v_F^2}\delta_0\Phi'\left(0,1\right). \tag{3.120}$$

Therefore, while the leading order scaling of the free energy is analytic T^3, the correction to scaling is non-analytic $T^{3+\eta_\delta}$.

3.6 Chiral symmetry breaking

In this Section we shall try to understand the physical nature of the instabilities of the symmetric phase of QED_3 or Algebraic Fermi Liquid.

We show that a phase disordered d-wave superconductor, as introduced in previous sections, becomes an antiferromagnet [83]. The antiferromagnetism emerges via a phenomenon of spontaneous chiral symmetry breaking (CSB)[59, 60]. Away from half filling, the broken symmetry phase typically takes the form of an incommensurate spin-density-wave (SDW), whose periodicity is tied to the Fermi surface. Furthermore, we show that numerous other states, most notably a d+ip and a d+is *phase-incoherent* superconductors (dipSC, disSC) and "stripes", i.e. superpositions of 1D charge-density-waves (CDW) and *phase-incoherent* superconducting-density-waves (SCDW), as well as continuous chiral rotations among them, are all energetically close and competitive with antiferromagnetism due to their equal membership in the chiral manifold of two-flavor ($N = 2$) QED_3. This large chiral manifold of nearly degenerate states plays the key role in the QED_3 theory as the culprit behind the complexity of the HTS phase diagram.

The above results place tight restrictions on this phase diagram and provide means to unify the phenomenology of cuprates within a single, systematically calculable "QED_3 Unified Theory" (QUT). Any *microscopic* description of cuprates, as long as it leads to the large d-wave pairing pseudogap with $T^* \gg T_c \to 0$, will conform to the general results of QUT. In particular, all the physical states in natural energetic proximity to a d-wave superconductor are the ones inhabiting the above chiral manifold.

For the sake of concreteness we restate the starting Lagrangian 3.47

$$\mathcal{L}_{\rm QED} = \bar{\psi}_n c_{\mu,n}\gamma_\mu D_\mu \psi_n + \mathcal{L}_0[a_\mu] + (\cdots). \tag{3.121}$$

Here $\psi^\dagger_\alpha = \bar{\psi}_\alpha\gamma_0 = (\eta^\dagger_\alpha, \eta^\dagger_{\bar{\alpha}})$ are the four-component Dirac spinors with $\eta^\dagger_\alpha = \frac{1}{\sqrt{2}}\Psi^\dagger_\alpha(\mathbf{1} + i\sigma_1)$, $\eta^\dagger_{\bar{\alpha}} = \frac{1}{\sqrt{2}}\Psi^\dagger_{\bar{\alpha}}\sigma_2(\mathbf{1} + i\sigma_1)$, and $\Psi^\dagger_\alpha = (\psi^\dagger_{\uparrow\alpha}, \psi_{\downarrow\alpha})$. Fermion fields $\psi_{\sigma\alpha}(\mathbf{r},\tau)$ describe 'topological fermions' of the theory and are related to the original nodal fermions $c_{\sigma\alpha}(\mathbf{r},\tau)$ via the singular gauge transformation (3.4). Index n labels $(1,\bar{1})$ and $(2,\bar{2})$ *pairs* of nodes while α labels individual nodes, $\mu = \tau, x, y (\equiv 0,1,2)$. $D_\mu = \partial_\mu + ia_\mu$ is a covariant derivative, $c_{\tau,n} = 1$, $c_{x,1} = c_{y,2} = v_F$, $c_{x,2} = c_{y,1} = v_\Delta$. The gamma matrices are defined as $\gamma_0 = \sigma_3 \otimes \sigma_3$, $\gamma_1 = -\sigma_3 \otimes \sigma_1$, $\gamma_2 = -\sigma_3 \otimes \sigma_2$, and satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The Berry gauge field a_μ encodes the topological frustration of nodal fermions generated by fluctuating quantum vortex-antivortex pairs and $\mathcal{L}_0$ is its bare action. The loss of superconducting phase coherence caused by unbinding of vortex pairs is heralded in (3.121) by a_μ becoming massless:

$$\mathcal{L}_0 \to \frac{1}{2e_\tau^2}(\partial\times a)^2_\tau + \sum_i \frac{1}{2e_i^2}(\partial\times a)^2_i \ ; \tag{3.122}$$

here $e^2_\tau, e^2_i (i = x, y)$, as well as the velocities $v_{F(\Delta)}$, are functions of doping x and T. Along with residual interactions between nodal fermions, denoted by the ellipsis in (3.121), these parameters of QUT arise from some more microscopic description and will be discussed shortly.

$\mathcal{L}_{\rm QED}$ (3.121) possesses the following peculiar continuous symmetry: borrowing from ordinary quantum electrodynamics in (3+1) dimensions ($\rm QED_4$), we know that there exist two additional gamma matrices, $\gamma_3 = \sigma_1 \otimes \mathbf{1}$ and $\gamma_5 = i\sigma_2 \otimes \mathbf{1}$ that anti-commute with *all* γ_μ. We can define a global U(2) symmetry for each pair of nodes, with generators $\mathbf{1}\otimes\mathbf{1}$, γ_3, $-i\gamma_5$ and $\frac{1}{2}[\gamma_3, \gamma_5]$, which leaves $\mathcal{L}_{\rm QED}$ invariant. In $\rm QED_3$ this symmetry can be broken by two "mass" terms, $m_{\rm ch}\bar{\psi}_n\psi_n$ and $m_{\rm PT}\bar{\psi}_n\frac{1}{2}[\gamma_3, \gamma_5]\psi_n$. Spontaneous symmetry breaking in $\rm QED_3$ as a mechanism for dynamical mass generation has been extensively studied in the field theory literature [59]. It has been established that while $m_{\rm PT}$ is never spontaneously generated [84], the chiral mass $m_{\rm ch}$ is generated if number of fermion species N is less than a critical value N_c. While

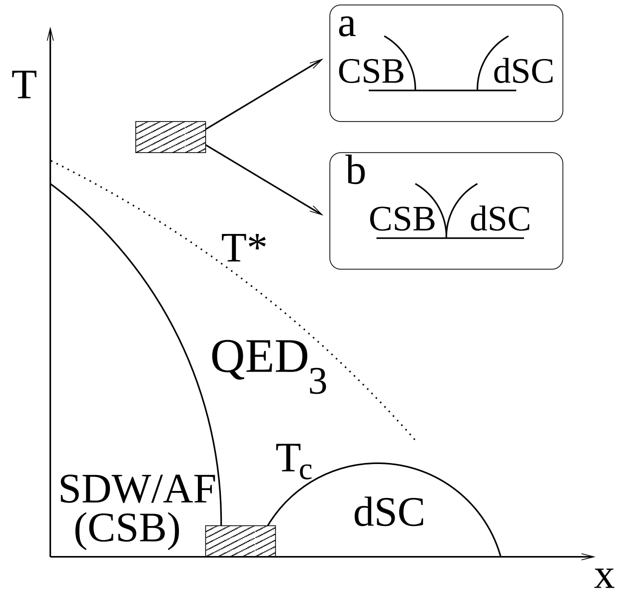


Figure 3.4: Schematic phase diagram of a cuprate superconductor in QUT. Depending on the value of N_c (see text), either the superconductor is followed by a *symmetric* phase of QED$_3$ which then undergoes a quantum CSB transition at some lower doping (panel a), or there is a direct transition from the superconducting phase to the $m_{\rm ch} \neq 0$ phase of QED$_3$ (panel b). The label SDW/AF indicates the dominance of the antiferromagnetic ground state as $x \to 0$.

still a matter of some controversy, standard non-perturbative methods give $N_c \sim 3$ for *isotropic* QED$_3$[59, 60], but as we shall discuss shortly, anisotropy and irrelevant couplings present in Lagrangian (3.121) can change the value of N_c.

Let us now *assume* that CSB occurs and the mass term $m_{\rm ch}\bar{\psi}_n\psi_n$ is generated. We wish to determine what is the nature of this chiral instability in terms of the original electron operators. To make this apparent, let us consider a general chiral rotation $\psi_n \to U_{\rm ch}^{(n)}\psi_n$ with $U_{\rm ch}^{(n)} = \exp(i\theta_{3n}\gamma_3 + \theta_{5n}\gamma_5)$. Within our representation of Dirac spinors (3.121), the $m_{\rm ch}\bar{\psi}_n\psi_n$ mass term takes the following form:

$$m_{\rm ch}\cos(2\Omega_n)\left[\eta_\alpha^\dagger\sigma_3\eta_\alpha - \eta_{\bar\alpha}^\dagger\sigma_3\eta_{\bar\alpha}\right]+ \\ +m_{\rm ch}\sin(2\Omega_n)\frac{\theta_{5n}+i\theta_{3n}}{\Omega_n}\eta_\alpha^\dagger\sigma_3\eta_{\bar\alpha} + \text{h.c.} \quad , \tag{3.123}$$

where $\Omega_n = \sqrt{\theta_{3n}^2 + \theta_{5n}^2}$. $m_{\rm ch}$ acts as an order parameter for the bilinear combinations of topological fermions appearing in (3.123). In the symmetric phase of QED$_3$ ($m_{\rm ch} =$

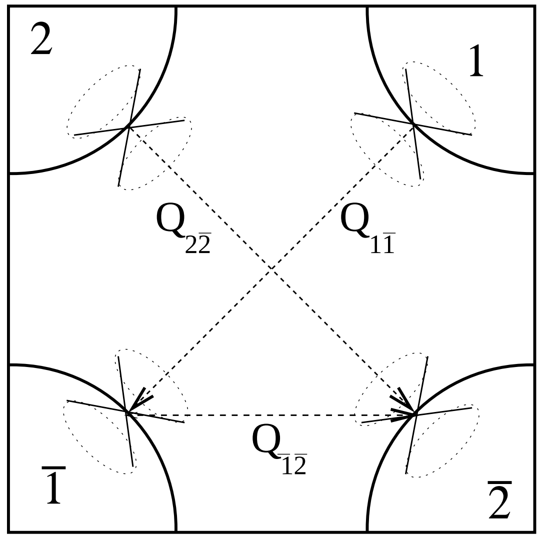


Figure 3.5: The "Fermi surface" of cuprates, with the positions of nodes in the d-wave pseudogap. The wavectors $\mathbf{Q}_{1\bar{1}}, \mathbf{Q}_{2\bar{2}}, \mathbf{Q}_{\bar{1}\bar{2}}$, etc. are discussed in the text.

0) the expectation values of such bilinears vanish, while they become finite, $\langle\bar{\psi}_n\psi_n\rangle \neq 0$, in the broken symmetry phase.

The chiral manifold (3.123) is spanned by the "basis" of three symmetry breaking states. When re-expressed in terms of the original nodal fermions $c_{\sigma\alpha}(\mathbf{r},\tau)$, two of these involve pairing in the particle-hole (p-h) channel – a cosine and a sine spin-density-wave (SDW):

$$\begin{aligned} &\langle c^\dagger_{\uparrow\alpha}c_{\uparrow\bar{\alpha}} - c^\dagger_{\downarrow\alpha}c_{\downarrow\bar{\alpha}}\rangle + \text{h.c.} \qquad &\text{(cos SDW)} \\ i&\langle c^\dagger_{\uparrow\alpha}c_{\uparrow\bar{\alpha}} - c^\dagger_{\uparrow\bar{\alpha}}c_{\uparrow\alpha}\rangle + (\uparrow\rightarrow\downarrow) \qquad &\text{(sin SDW)} \end{aligned} \tag{3.124}$$

and are obtained from Eq. (3.123) by setting Ω_n equal to $\pi/4$ or $3\pi/4$. Rotations within the chiral manifold (3.123) at fixed Ω_n correspond to the sliding modes of SDW.

A simple physical picture emerges here: we started from a d-wave superconducting phase, our parent state. As one moves closer to half-filling and true phase coherence is lost, strong vortex-antivortex pair fluctuations, acting under the protective umbrella of a d-wave particle-particle (p-p) pseudogap, spontaneously induce formation of particle-hole "pairs" at finite wavevectors $\pm\mathbf{Q}_{1\bar{1}}$ and $\pm\mathbf{Q}_{2\bar{2}}$, spanning the Fermi surface from node α to $\bar{\alpha}$ (Fig. 3.5). The glue that binds these p-h "pairs" and plays the role of "phonons" in this pairing analogy is provided by the Berry gauge field a_μ. Such "fermion duality" is a natural consequence of the QED_3 theory (3.121). Remarkably, we find the antiferromagnetic insulator being spontaneously generated in form of the incommensurate SDW. As we get very near half-filling and $\mathbf{Q}_{1\bar{1}}, \mathbf{Q}_{2\bar{2}}$ approach $(\pm\pi, \pm\pi)$, SDW acquires the most favored state status within the chiral manifold – this is the consequence of umklapp processes which increase its condensation energy without it being offset by either the anisotropy or a poorly screened Coulomb interaction which plagues its CDW competitors to be introduced shortly. It seems therefore reasonable to argue that this SDW must be considered the progenitor of the Neel-Mott-Hubbard insulating antiferromagnet at half-filling. Thus, QED_3 theory (3.121) explains the origin of antiferromagnetic order in terms of strong vortex-antivortex fluctuations in the parent d-wave superconductor. It does so naturally, through its inherent and well-established chiral symmetry breaking instability [59].

The chiral manifold (3.123) contains also a third state, a p-p pairing state corresponding to $\Omega_n = 0$ or $\pi/2$ and best characterized as a d+ip phase-incoherent superconductor:

$$i\langle\psi_{\uparrow\alpha}\psi_{\downarrow\alpha} - \psi_{\uparrow\bar{\alpha}}\psi_{\downarrow\bar{\alpha}}\rangle + \text{h.c.} \qquad\qquad (\text{dipSC})\ . \qquad (3.125)$$

We have written dipSC in terms of topological fermions $\psi_{\sigma\alpha}(\mathbf{r}, \tau)$ since use of the original fermions leads to more complicated expression which involves the backflow of vortex-antivortex excitations described by gauge fields a_μ and v_μ (such backflow terms do not arise in the p-h channel). This state breaks parity but preserves time reversal, translational invariance and superconducting U(1) symmetries. To our knowledge, such state has not been proposed as a part of any of the major theories of HTS. It is an intriguing question whether this d+ip phase-incoherent superconductor can be the actual ground state at some dopings in some of the cuprates. Its energetics does not suffer from long range Coulomb problems but it is clearly inferior to the SDW very close to half-filling since, being spatially uniform, it receives no help from umklapp.

Until now, we have discussed the CSB pattern only within individual pairs of nodes, (1,$\bar{1}$) and (2,$\bar{2}$). What happens if we allow for chiral rotations that mix nodes

1 and $\bar{2}$ or 1 and 2? A whole new plethora of states becomes possible, with chiral manifold enlarged to include a superposition of *one-dimensional* p-h and p-p states, an incommensurate CDW accompanied by a non-uniform phase-incoherent superconductor (SCDW) at wavevectors $\pm\mathbf{Q}_{12}$ and $\pm\mathbf{Q}_{\bar{2}\bar{1}}$ (Fig. 3.5):

$$\frac{1}{\sqrt{2}}\langle c^\dagger_{\uparrow 1}c_{\uparrow 2} + c^\dagger_{\uparrow\bar{2}}c_{\uparrow\bar{1}} + \text{h.c.}\rangle + (\uparrow\rightarrow\downarrow) \quad \text{(CDW)}$$

$$\frac{1}{\sqrt{2}}\langle \psi_{\uparrow 1}\psi_{\downarrow 2} + \psi_{\uparrow\bar{2}}\psi_{\downarrow\bar{1}} + \text{h.c.}\rangle + (\uparrow\leftrightarrow\downarrow) \quad \text{(SCDW)} \qquad (3.126)$$

These same states, rotated by $\pi/2$, are replicated at wavevectors $\pm\mathbf{Q}_{1\bar{2}}$ and $\pm\mathbf{Q}_{2\bar{1}}$ (Fig. 2). In a fluctuating $d_{x^2-y^2}$ superconductor these CDWs and SCDWs run along the x and y axes and are naturally identified as the "stripes" of QUT. Note, however, these are not the only one-dimensional states in QUT – among the states in the chiral manifold (3.123) are also "diagonal stripes", the combination of a SDW (3.124) along $\pm\mathbf{Q}_{1\bar{1}}$ and a dipSC (3.125) which opens the mass gap only at nodes $(2,\bar{2})$, or vice versa. Furthermore, a phase-incoherent d+is superconductor (disSC) is also present within the chiral enlarged manifold, since it results in alternating signs for different nodes with equal number of positive and negative "masses" for the two-component nodal fermions:

$$i\langle \psi_{\uparrow 1}\psi_{\downarrow 1} + \psi_{\uparrow\bar{1}}\psi_{\downarrow\bar{1}} + \text{h.c.}\rangle + (1\rightarrow 2) \qquad \text{(disSC)}\ . \qquad (3.127)$$

In contrast, in a d+id phase incoherent superconductor these "masses" have the same sign for all the nodes producing a maximal breaking of the PT symmetry [84]. Consequently, a d+id phase-incoherent superconductor is not spontaneously induced within the QED_3 theory.

In the *isotropic* ($v_F = v_\Delta$) $N = 2$ QED_3 all these additional states plus arbitrary *chiral* rotations among them are completely equivalent to those discussed previously. It is here where we confront the problem of intrinsic anisotropy in Eq. (3.121). Such anisotropy cannot be rescaled out and manifestly breaks the U(2)×U(2) degeneracy of the full $N = 2$ chiral manifold down to two separate U(1)×U(1) (3.123) chiral groups discussed previously. This is reflected in the general increase in energy of the states from the enlarged chiral manifold. For example, the anisotropy raises the energy of our “stripe” states (3.126) relative to those of SDW, dipSC or “diagonal stripes”. However, when the long range Coulomb interactions and coupling to the lattice are included in the problem, as they are in real materials, it is conceivable that the “stripes” would return in some form, either as a ground state or a long-lived metastable state at some intermediate doping. disSC is also adversely affected by anisotropy but to a lesser extent and might remain competitive with SDW, dipSC and “diagonal stripes”. This state breaks time reversal symmetry but preserves parity and the discussion concerning dipSC below Eq. (3.125) applies to disSC equally well.

How do we use these general results on CSB in QUT to address the specifics of cuprate phase diagram? To this end, we need some effective combination of phenomenology and more microscopic descriptions to determine the parameters v_F, v_Δ, e_τ, e_i and residual interactions $(\cdots)$ appearing in $\mathcal{L}_{\rm QED}$ (3.121). The main task is to determine what is the sequence of states within QUT that form stable phases as the doping decreases toward half-filling under T^* in Fig. 3.4. While this is an extensive project for the future, we outline here some of the general features. First, within the superconducting state $e_\tau, e_i \to 0$ and a_μ becomes massive thus denying the CSB mechanism its main dynamical agent. We therefore expect that the superconductor is in the symmetric phase and its nodal fermions form well-defined excitations[58]. As we move to the left in Fig. 3.4, the phase order is suppressed and e_τ, e_i become finite, reflecting the unbinding of vortex-antivortex excitations[58]. For all practical purposes, this is precisely what the experiments imply. Now, the key question is whether the QED$_3$ (3.121) remains in its symmetric phase or whether it immediately undergoes the CSB transition and generates finite gap ($m_{\rm ch} \neq 0$).

This question is difficult to answer in detail, because N_c can depend non-universally on higher order coupling constants. For instance, short range interactions, while perturbatively irrelevant, effectively increase N_c if stronger than some critical value[74]. Such interactions, typically in the form of short range three-current terms[21], arise in more microscopic models used to derive $\mathcal{L}_{\rm QED}$ and are prominent among the residual terms denoted by ellipsis in (3.121). Their strength generically increases as $x \to 0$. These residual interactions play a dual role in QUT. First, they can conspire with the anisotropy to produce the situation depicted in panel (b) of Fig. 3.4, where the CSB takes place as soon as the phase coherence is lost. Second, once the chiral symmetry has been broken, the residual interactions further break the symmetry within the chiral manifold (3.123) and play a role in selecting the true ground state.

3.7 Summary and conclusions

We considered the effect of fluctuations in the d-wave superconductor, concentrating on the role of the phase fluctuations and their effect on the nodal fermions. The pairing pseudogap provided a reference point of our analysis which allowed us to classify various quasiparticle interactions according to their fate under the renormalization group. By carefully treating the interactions of the quasiparticles with the vortex phase defects, we find that the low energy effective theory for the quasiparticles inside the pairing protectorate is the (2+1) dimensional quantum electrodynamics (QED_3) with inherent spatial anisotropy, described by Lagrangian $\mathcal{L}_D$ specified by Eqs. (3.47,3.57).

Within the superconducting state the gauge fields of the theory are massive by virtue of vortex defects being bound into finite loops or vortex-antivortex pairs. Such massive gauge fields produce only short ranged interactions between our BdG quasiparticles and are therefore irrelevant: in the superconductor quasiparticles remain sharp in agreement with prevailing experimental data [85]. Loss of the long range superconducting order is brought about by unbinding the topological defects – vortex loops or vortex-antivortex pairs – via Kosterlitz-Thouless type transition and its quantum analogue. Remarkably, this is accompanied by the Berry gauge field becoming massless. Such massless gauge field mediates long range interactions between the fermions and becomes a relevant perturbation. Exactly what is the consequence of this relevant perturbation depends on the number of fermion species N in the problem. For cuprates we argued that $N = 2n_{\mathrm{CuO}}$ where n_{CuO} is the number of CuO_2 layers per unit cell. If $N < N_c \simeq 3$ [59], the interactions cause spontaneous opening of a gap for the fermionic excitations at $T = 0$, via the mechanism of chiral symmetry breaking in QED_3 [59]. Formation of the gap corresponds to the onset of AF SDW instability [58, 83] which must be considered as a progenitor of the Mott-Hubbard-

Neel antiferromagnet at half-filling. If, on the other hand, $N > N_c$ as will be the case in bilayer or trilayer materials, the theory remains in its chirally symmetric non-superconducting phase even as $T \to 0$ and AF order arises from within such state only upon further underdoping (Fig. 1). We call this symmetric state of QED_3 an algebraic Fermi liquid (AFL). In both cases AFL controls the low temperature, low energy behavior of the pseudogap state and in this sense assumes the role played by Fermi liquid theory in conventional metals and superconductors. In AFL the quasi-particle pole is replaced by a branch cut – the quasiparticle is no longer sharp – and the fermion propagator acquires a Luttinger-like form Eq. (3.68). To our knowledge this is one of the very few cases where a non-Fermi liquid nature of the excitations has been demonstrated in dimension greater than one in the absence of disorder or magnetic field. This Luttinger-like behavior of AFL will manifest itself in anomalous power law functional form of many physical properties of the system.

Dirac anisotropy, i.e. the fact that $v_F \neq v_\Delta$, plays important role in the cuprates where the ratio $\alpha_D = v_F/v_\Delta$ in most materials ranges between 3 and 15−20. Such anisotropy is nontrivial as it cannot be rescaled and it significantly complicates any calculation within the theory. Using the perturbative renormalization group theory we have shown that anisotropic QED_3 flows back into isotropic stable fixed point. This means that for weak anisotropy at long lengthscales the universal properties of the theory are identical to those of the simple isotropic case. It remains to be seen what are the properties of the theory at intermediate lengthscales when anisotropy is strong.

The QED_3 theory of the pairing pseudogap, as presented in here, starts from a remarkably simple set of assumptions, and via manipulations that are controlled in the sense of $1/N$ expansion, arrives at nontrivial consequences, including the algebraic Fermi liquid and the antiferromagnet.

Appendix A

Quasiparticle correlation functions in the mixed state

A.1 Green's functions: definitions and identities

The one-particle Green's function matrix [52] is defined as

$$\hat{G}_{\alpha\beta}(\mathbf{r}_1\tau_1;\mathbf{r}_2\tau_2) \equiv -\langle T_\tau \Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle \tag{A.1}$$

where T_τ denotes imaginary time ordering operator and $\alpha = 1,2$ denotes components of a Nambu spinor $\Psi^\dagger(1)$ which is a shorthand for $\Psi^\dagger(\mathbf{r}_1,\tau_1) = \big(\psi^\dagger_\uparrow(\mathbf{r}_1,\tau_1), \psi_\downarrow(\mathbf{r}_1,\tau_1)\big)$.

Due to the time independence of the Hamiltonian (2.2), the Green's function (A.1) depends only on the imaginary time difference $\tau = \tau_1 - \tau_2$. Therefore, its Fourier transform is given by

$$\begin{aligned}
\hat{G}(\mathbf{r}_1,\mathbf{r}_2;i\omega) &= \int_0^\beta e^{i\omega\tau}\hat{G}(\mathbf{r}_1,\mathbf{r}_2,\tau)d\tau \\
\hat{G}(\mathbf{r}_1,\mathbf{r}_2;\tau) &= \frac{1}{\beta}\sum_{i\omega} e^{-i\omega\tau}\hat{G}(\mathbf{r}_1,\mathbf{r}_2;i\omega).
\end{aligned} \tag{A.2}$$

Here $\beta = 1/(k_B T)$, T is temperature, and the fermionic frequency $\omega = (2l+1)\pi/\beta$, $l \in Z$.

Using the above relations, it is straightforward to derive the spectral representation of the following correlation functions between Ψ and its imaginary time derivatives

$\partial_\tau \Psi \equiv \dot{\Psi}$:

$$\begin{cases} \langle T_\tau \Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle \\ \langle T_\tau \dot{\Psi}_\alpha(1)\Psi^\dagger_\beta(2)\rangle \\ \langle T_\tau \Psi_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle \\ \langle T_\tau \dot{\Psi}_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle \end{cases} = -\frac{1}{\beta}\sum_{i\omega} e^{-iw\tau}\int d\epsilon \frac{\hat{A}(\mathbf{r}_1,\mathbf{r}_2;\epsilon)}{i\omega-\epsilon} \begin{cases} +1 \\ -\epsilon \\ +\epsilon \\ -\epsilon^2 \end{cases} \tag{A.3}$$

The spectral function $\hat{A}(\mathbf{r}_1,\mathbf{r}_2;\epsilon)$ can be written in terms of eigenfunctions $\Phi_n(\mathbf{r}) = (u_n(\mathbf{r}), v_n(\mathbf{r}))^T$ of the Bogoliubov-deGennes Hamiltonian $\hat{H}_0$ in the form:

$$\hat{A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\epsilon) = \sum_n \delta(\epsilon-\epsilon_n)\Phi_{n\alpha}(\mathbf{r}_1)\Phi^\dagger_{n\beta}(\mathbf{r}_2), \tag{A.4}$$

where ϵ_n is the eigen-energy associated with an eigenstate labeled by the quantum number n. Substituting (A.4) into (A.3) we can write the above correlation functions solely in terms of the eigenfunctions $\Phi_n(\mathbf{r})$:

$$\begin{cases} \langle T_\tau \Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle \\ \langle T_\tau \dot{\Psi}_\alpha(1)\Psi^\dagger_\beta(2)\rangle \\ \langle T_\tau \Psi_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle \\ \langle T_\tau \dot{\Psi}_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle \end{cases} = -\frac{1}{\beta}\sum_{i\omega,n} e^{-iw\tau}\frac{\Phi_{n\alpha}(\mathbf{r}_1)\Phi^\dagger_{n\beta}(\mathbf{r}_2)}{i\omega-\epsilon_n} \begin{cases} +1 \\ -\epsilon_n \\ +\epsilon_n \\ -\epsilon_n^2 \end{cases} \tag{A.5}$$

In the calculations that follow we will also encounter Matsubara summations over the fermionic frequencies $\omega = (2l+1)\pi/\beta$, $l \in Z$, of the form:

$$S_{nm}(i\Omega) = \frac{1}{\beta}\sum_{i\omega}\frac{1}{(i\omega-\epsilon_n)(i\Omega+i\omega-\epsilon_m)} \tag{A.6}$$

where $\Omega = 2\pi k/\beta$, $k \in Z$, is an outside bosonic frequency. The sum (A.6) can be evaluated using standard techniques (see e.g. [52]) and yields:

$$S_{nm}(i\Omega) = \frac{f_n - f_m}{\epsilon_n - \epsilon_m + i\Omega}. \tag{A.7}$$

Here f_n is a short hand for the Fermi-Dirac distribution function $f(\epsilon_n) = \big(1 + \exp(\beta\epsilon_n)\big)^{-1}$.

In order to derive spin and thermal currents we will need the explicit form of the generalized velocity operator $\mathbf{V}$ introduced in (2.33):

$$\mathbf{V} = \begin{pmatrix} \frac{\boldsymbol{\pi}}{m} & i\hat{\mathbf{v}}_\Delta \\ i\hat{\mathbf{v}}^*_\Delta & \frac{\boldsymbol{\pi}^*}{m} \end{pmatrix} \tag{A.8}$$

where the gap velocity operator is given by

$$i\hat{\mathbf{v}}_\Delta = i\Delta_0\eta_{\boldsymbol{\delta}}\boldsymbol{\delta}e^{i\phi(\mathbf{r})/2}\left(e^{i\boldsymbol{\delta}\cdot\mathbf{p}} - e^{-i\boldsymbol{\delta}\cdot\mathbf{p}}\right)e^{i\phi(\mathbf{r})/2} \tag{A.9}$$

and the canonical momentum equals

$$\boldsymbol{\pi} = -\frac{i}{2}\boldsymbol{\delta}e^{\frac{i}{\hbar}\int_{\mathbf{r}}^{\mathbf{r}+\delta}(\mathbf{p}-\frac{e}{c}\mathbf{A})\cdot d\mathbf{l}} + h.c. \tag{A.10}$$

The following identities for operators $\hat{\mathbf{v}}_\Delta$ and $\boldsymbol{\pi}$ will be used in the next section:

$$\begin{cases} \boldsymbol{\pi}^*\Psi^\dagger\cdot\boldsymbol{\pi}\Psi &= \quad i\hbar\nabla\cdot(\Psi^\dagger\boldsymbol{\pi}\Psi) + \Psi^\dagger\boldsymbol{\pi}^2\Psi \\ \boldsymbol{\pi}^*\Psi^\dagger\cdot\boldsymbol{\pi}\Psi &= -i\hbar\nabla\cdot(\boldsymbol{\pi}^*\Psi^\dagger\ \ \Psi) + (\boldsymbol{\pi}^*)^2\Psi^\dagger\ \ \Psi \end{cases} \tag{A.11}$$

$$\Psi^\dagger\Delta\Psi - \Delta\Psi^\dagger\ \ \Psi = \frac{1}{2}\nabla\cdot\left(\Psi^\dagger\ \ \hat{\mathbf{v}}_\Delta\Psi - \hat{\mathbf{v}}_\Delta\Psi^\dagger\ \ \Psi\right) \tag{A.12}$$

The above equations are straightforward to derive in continuum, while on the tight-binding lattice Eqs. (A.11) and (A.12) imply a symmetric definition of the lattice divergence operator.

The identities (A.11,A.12) explicitly ensure that the generalized velocity operator V_μ is Hermitian i.e it satisfies the following identity:

$$\int d\mathbf{r}\mathbf{V}^*\Psi^\dagger(\mathbf{r})\ \ \Psi(\mathbf{r}) \equiv \int d\mathbf{r}\mathbf{V}^*_{\alpha\beta}\Psi^\dagger_\beta(\mathbf{r})\ \ \Psi_\alpha(\mathbf{r})$$
$$= \int d\mathbf{r}\Psi^\dagger(\mathbf{r})\ \ \mathbf{V}\Psi(\mathbf{r}). \tag{A.13}$$

A.2 Spin current and spin conductivity tensor

The time derivative of spin density $\rho_s = \frac{\hbar}{2}(\psi^\dagger_\uparrow\psi_\uparrow - \psi^\dagger_\downarrow\psi_\downarrow)$ can be written in Nambu formalism as

$$\dot{\rho}^s = \frac{i}{\hbar}[H,\rho^s] = \frac{\hbar}{2}(\dot{\Psi}^\dagger\Psi + \Psi^\dagger\dot{\Psi}) \tag{A.14}$$

Using equations of motion (2.10) together with the explicit form of $\hat{H}_0$ operator (2.8) we obtain

$$\dot{\rho}^s = \frac{i}{2}\left(\frac{(\pi^*)^2}{2m}\Psi^\dagger_1\ \ \Psi_1 - \Psi^\dagger_1\frac{\pi^2}{2m}\Psi_1 + \hat{\Delta}\Psi^\dagger_1\ \ \Psi_2 - \Psi^\dagger_1\hat{\Delta}\Psi_2\right.$$
$$\left. + \hat{\Delta}^*\Psi^\dagger_2\ \ \Psi_1 - \Psi^\dagger_2\ \ \hat{\Delta}^*\Psi_1 - \frac{\pi^2}{2m}\Psi^\dagger_2\ \ \Psi_2 + \Psi^\dagger_2\frac{(\pi^*)^2}{2m}\Psi_2\right). \tag{A.15}$$

Using the identities (A.11,A.12) it is easy to show that

$$\dot{\rho}^s = -\frac{\hbar}{4}\nabla_\mu(\Psi^\dagger V_\mu \Psi + V_\mu^* \Psi^\dagger \ \ \Psi) = -\nabla \cdot \mathbf{j}^s \tag{A.16}$$

where the generalized velocity operator V_μ is defined in Eq. (A.8), and the last equality follows from the continuity equation (2.31) relating the spin density ρ^s and the spin current $\mathbf{j}^s$. Upon spatial averaging and utilizing Eq. (A.13) we find the $q \to 0$ limit of the spin current

$$j_\mu^s = \frac{\hbar}{2}\Psi^\dagger V_\mu \Psi. \tag{A.17}$$

The evaluation of the spin current-current correlation function (2.29) is straightforward and yields:

$$D_{\mu\nu}(i\Omega) = -\frac{\hbar^2}{4}\int_0^\beta e^{i\tau\Omega}\Big[V_\mu(2)V_\nu(4)\times \langle T_\tau \Psi^\dagger(1)\Psi(2)\Psi^\dagger(3)\Psi(4)\rangle_{\substack{1,2\to(\mathbf{x},\tau)\\ 3,4\to(\mathbf{y},0)}}\Big]d\tau \tag{A.18}$$

Here $\Psi(1) \equiv \Psi(\mathbf{r}_1, \tau_1)$ and similarly the operator $V_\mu(2)$ acts only on functions of $\mathbf{r}_2$. Using Wick's theorem, identity (A.5), and upon spatial averaging over $\mathbf{x}$ and $\mathbf{y}$ we obtain

$$D_{\mu\nu}(i\Omega) = \frac{\hbar^2}{4}\sum_{mn}\langle n|V_\mu|m\rangle\langle m|V_\nu|n\rangle S_{nm}(i\Omega). \tag{A.19}$$

The double summation extends over the eigenstates $|m\rangle$ and $|n\rangle$ of the Hamiltonian (2.8), and $S_{mn}(i\Omega)$ is given in Eq. (A.7). Analytically continuing $i\Omega \to \Omega + i0$ we finally obtain the expression for the retarded correlation function:

$$D_{\mu\nu}^R(\Omega) = \frac{\hbar^2}{4}\sum_{mn}\frac{V_\mu^{nm}V_\nu^{mn}}{\epsilon_n - \epsilon_m + \Omega + i0}(f_n - f_m). \tag{A.20}$$

where V_{mn}^μ is defined in (2.35) and f_n is a short hand for the Fermi-Dirac distribution function $f(\epsilon_n) = \big(1+\exp(\beta\epsilon_n)\big)^{-1}$. Finally, we substitute the last equation into (2.28) to obtain

$$\sigma_{\mu\nu}^s = \frac{\hbar^2}{4i}\sum_{mn}\frac{V_\mu^{nm}V_\nu^{mn}}{(\epsilon_n - \epsilon_m + i0)^2}(f_n - f_m) \tag{A.21}$$

A.3 Thermal current and thermal conductivity tensor

In order to calculate thermal currents and thermal conductivity in the magnetic field we introduce a pseudo-gravitational potential $\chi = \mathbf{r}\cdot\mathbf{g}/c^2$ [48, 49, 50, 51]. This formal procedure is useful because it illustrates that the transverse thermal response is not given just by Kubo formula, but in addition it includes corrections related to magnetization. Throughout this section $\hbar = 1$. The pseudo-gravitational potential enters the Hamiltonian, up to linear order in χ, as

$$H_T = \int d\mathbf{x}\ \ (1+\frac{\chi}{2})\Psi^\dagger(\mathbf{x})\ \hat{H}_0\ (1+\frac{\chi}{2})\Psi(\mathbf{x}), \tag{A.22}$$

where H_0 is the Bogoliubov-deGennes Hamiltonian (2.8). The equations of motion for the fields Ψ thus become

$$\begin{aligned} i\dot{\Psi} = [\Psi, H_T] &= (1+\frac{\chi}{2})\hat{H}_0(1+\frac{\chi}{2})\Psi \\ &= (1+\chi)\hat{H}_0\Psi - i\nabla_\mu\chi V_\mu\Psi. \end{aligned} \tag{A.23}$$

The last equality follows from the commutation relation (2.33). (Note that for $\chi \neq 0$ the Eq. (A.23) differs from Eq. (2.10). Throughout this section $\dot{\Psi}$ will refer to Eq. (A.23) unless explicitly stated otherwise).

To find thermal current $\mathbf{j}^Q$ we start with the continuity equation

$$\dot{h}_T + \nabla\cdot\mathbf{j}^Q = 0. \tag{A.24}$$

The Hamiltonian density h_T follows from Eq. (A.22) and reads

$$\begin{aligned} h_T = \frac{1}{2m^*}\left(\pi^*_\mu\tilde{\Psi}^\dagger_1\ \ \pi_\mu\tilde{\Psi}_1 - \pi_\mu\tilde{\Psi}^\dagger_2\ \ \pi^*_\mu\tilde{\Psi}_2\right) - \mu\tilde{\Psi}^\dagger_\alpha\sigma^3_{\alpha\beta}\tilde{\Psi}_\beta \\ + \frac{1}{2}\left(\tilde{\Psi}^\dagger_\alpha\Delta_{\alpha\beta}\tilde{\Psi}_\beta + \Delta_{\alpha\beta}\tilde{\Psi}^\dagger_\alpha\ \ \tilde{\Psi}_\beta\right) \end{aligned} \tag{A.25}$$

where $\tilde{\Psi} = (1+\frac{\chi}{2})\Psi$. Taking the time derivative of the Hamiltonian density h_T and using Eqs.(A.11) and equations of motion (A.23) we obtain

$$\dot{h}_T = i[H_T, h_T] = \frac{i}{2m}\nabla_\mu \left(\dot{\tilde{\Psi}}^\dagger \Pi_\mu \;\; \tilde{\Psi} - \Pi^*_\mu \tilde{\Psi}^\dagger \;\; \dot{\tilde{\Psi}}\right)$$
$$+ \frac{1}{2}\left(\tilde{\Psi}_\alpha \Delta_{\alpha\beta}\dot{\tilde{\Psi}}_\beta - \dot{\tilde{\Psi}}^\dagger_\alpha \Delta_{\alpha\beta}\tilde{\Psi}_\beta\right)$$
$$+ \frac{1}{2}\left(\Delta_{\alpha\beta}\dot{\tilde{\Psi}}^\dagger_\alpha \;\; \tilde{\Psi}_\beta - \Delta_{\alpha\beta}\tilde{\Psi}^\dagger_\alpha \;\; \dot{\tilde{\Psi}}_\beta\right) \quad (A.26)$$

where we introduced matrix operators

$$\Delta_{\alpha\beta} = \begin{pmatrix} 0 & \hat{\Delta} \\ \hat{\Delta}^* & 0 \end{pmatrix}, \qquad \Pi^\mu_{\alpha\beta} = \begin{pmatrix} \pi_\mu & 0 \\ 0 & \pi^*_\mu \end{pmatrix}. \quad (A.27)$$

Here $\dot{\tilde{\Psi}} = (1 + \frac{\chi}{2})\dot{\Psi}$ and π_μ is defined in Eq. (A.10). Finally we use (A.12) to extract $\mathbf{j}^Q$ from (A.26). Upon spatial averaging and using the Hermiticity of V_μ (A.13) the thermal current $\mathbf{j}^Q$ reads

$$j^Q_\mu = \frac{i}{2}\left(\tilde{\Psi}^\dagger V_\mu \dot{\tilde{\Psi}} - \dot{\tilde{\Psi}}^\dagger V_\mu \tilde{\Psi}\right). \quad (A.28)$$

Note that the expression for $\mathbf{j}^Q$ contains two terms

$$\mathbf{j}^Q = \mathbf{j}^Q_0 + \mathbf{j}^Q_1 \quad (A.29)$$

where $\mathbf{j}^Q_0$ is independent of χ and $\mathbf{j}^Q_1$ is linear in χ. Explicitly:

$$\mathbf{j}^Q_0(\mathbf{r}) = \frac{1}{2}\Psi^\dagger\{\mathbf{V}, \hat{H}_0\}\Psi \quad (A.30)$$

and

$$j^Q_{\mu,1}(\mathbf{r}) = -\frac{i}{4}\partial_\nu\chi\Psi^\dagger\left(V_\mu V_\nu - V_\nu V_\mu\right)\Psi$$
$$+ \frac{\partial_\nu\chi}{4}\Psi^\dagger\left((x_\nu V_\mu + 3V_\mu x_\nu)\hat{H}_0 + \hat{H}_0(3x_\nu V_\mu + V_\mu x_\nu)\right)\Psi \quad (A.31)$$

where $\{a, b\} = ab+ba$. Analogously to the situation in the normal metal, the thermal average of $\mathbf{j}^Q_1$ does not in general vanish in the presence of the magnetic field [54].

The linear response of the system to the external perturbation can be described by

$$\langle j^Q_\mu\rangle = \langle j^Q_{0\mu}\rangle + \langle j^Q_{1\mu}\rangle = -(K_{\mu\nu} + M_{\mu\nu})\partial_\nu\chi \equiv -L^Q_{\mu\nu}\partial_\nu\chi \quad (A.32)$$

where

$$K_{\mu\nu} = -\frac{\delta \langle j^Q_{0\mu} \rangle}{\delta\, \partial_\nu \chi} = -\lim_{\Omega \to 0} \frac{P^R_{\mu\nu}(\Omega) - P^R_{\mu\nu}(0)}{i\Omega} \tag{A.33}$$

is the standard Kubo formula for a dc response [52], $P^R_{\mu\nu}(\Omega)$ being the retarded current-current correlation function, and

$$M_{\mu\nu} = -\frac{\delta \langle j^Q_{1\mu} \rangle}{\delta\, \partial_\nu \chi}$$
$$= -\sum_n \epsilon_n f_n \langle n| \{V_\mu, x_\nu\} |n\rangle + \sum_n \frac{i}{4} f_n \langle n|[V_\mu, V_\nu]|n\rangle \tag{A.34}$$

is a contribution from "diathermal" currents [54]. Note that the latter vanishes for the longitudinal response while it remains finite for the transverse response. As we will show later in this section, at $T = 0$ there is an important cancellation between (A.33) and (A.34) which renders the thermal conductivity $\kappa_{\mu\nu}$ well-behaved and prevents the singularity from the temperature denominator in Eq. (2.48).

The retarded thermal current-current correlation function $P^R_{\mu\nu}(\Omega)$ can be expressed in terms of the Matsubara finite temperature correlation function

$$\mathcal{P}_{\mu\nu}(i\Omega) = -\int_0^\beta d\tau e^{i\Omega\tau} \Big\langle T_\tau j_\mu(\mathbf{r}, \tau) j_\nu(\mathbf{r}', 0) \Big\rangle \tag{A.35}$$

as

$$P^R_{\mu\nu}(\Omega) = \mathcal{P}_{\mu\nu}(i\Omega \to \Omega + i0). \tag{A.36}$$

The current $j_\mu(\mathbf{r}, \tau)$ in Eq. (A.35) is given by

$$j_\mu(\tau) = \frac{1}{2}(\Psi^\dagger(\tau) V_\mu \partial_\tau \Psi(\tau) - \partial_\tau \Psi^\dagger(\tau)\ \ V_\mu \Psi(\tau)) \tag{A.37}$$

As pointed out by Ambegaokar and Griffin [56] time ordering operator T_τ and time derivative operators ∂_τ do not commute and neglecting this subtlety can lead to formally divergent frequency summations[38]. Taking heed of this subtlety, substitution of (A.37) into (A.35) amounts to

$$\mathcal{P}_{\mu\nu}(i\Omega) = -\frac{1}{4} \int_0^\beta d\tau e^{i\Omega\tau} V^\mu_{\alpha\beta}(2) V^\nu_{\gamma\delta}(4) \Big\langle T_\tau \dot{\Psi}^\dagger_\alpha(1) \Psi_\beta(2) \dot{\Psi}^\dagger_\gamma(3) \Psi_\delta(4)$$
$$- \dot{\Psi}^\dagger_\alpha(1) \Psi_\beta(2) \Psi^\dagger_\gamma(3) \dot{\Psi}_\delta(4) - \Psi^\dagger_\alpha(1) \dot{\Psi}_\beta(2) \dot{\Psi}^\dagger_\gamma(3) \Psi_\delta(4) + \Psi^\dagger_\alpha(1) \dot{\Psi}_\beta(2) \Psi^\dagger_\gamma(3) \dot{\Psi}_\delta(4) \Big\rangle, \tag{A.38}$$

where the notation follows Eq. (A.18) and Eq. (2.10). Upon utilizing the identities (A.5) and (A.13) and performing the standard Matsubara summation we have

$$\mathcal{P}_{\mu\nu}(i\Omega) = \frac{1}{4}\sum_{nm}\langle n|V_\mu|m\rangle\langle m|V^\nu|n\rangle(\epsilon_n+\epsilon_m)^2 S_{nm}(i\Omega) \tag{A.39}$$

where S_{mn} is given by Eq. (A.7). Analytically continuing $i\Omega \to \Omega + i0$ we obtain

$$\mathcal{P}^R_{\mu\nu}(\Omega) = \frac{1}{4}\sum_{nm}\frac{(\epsilon_n+\epsilon_m)^2\, V_\mu^{nm}V_\nu^{mn}}{\epsilon_n-\epsilon_m+\Omega+i0}(f_n-f_m). \tag{A.40}$$

Note that the only difference between the Kubo contribution to the thermal response (A.40) and the spin response (A.20) is the value of the coupling constant. In the case of thermal response (A.40), the coupling constant is $(\epsilon_n+\epsilon_m)/2$ which is eigenstate dependent, while in the case of spin response (A.20) it is $\hbar/2$ and eigenstate independent.

Using Eq. (A.33) we find that the Kubo contribution to the thermal transport coefficient is given by

$$K_{\mu\nu} = -\frac{i}{4}\sum_{nm}\frac{(\epsilon_n+\epsilon_m)^2}{(\epsilon_n-\epsilon_m+i0)^2}V_\mu^{nm}V_\nu^{mn}(f_n-f_m). \tag{A.41}$$

This can be written as

$$K_{\mu\nu} = K^{(1)}_{\mu\nu} + K^{(2)}_{\mu\nu} \tag{A.42}$$

where

$$K^{(1)}_{\mu\nu} = -\frac{i}{4}\sum_{nm}\frac{4\epsilon_n\epsilon_m}{(\epsilon_n-\epsilon_m+i0)^2}V_\mu^{nm}V_\nu^{mn}(f_n-f_m) \tag{A.43}$$

and

$$K^{(2)}_{\mu\nu} = -\frac{i}{4}\sum_{nm}V_\mu^{nm}V_\nu^{mn}(f_n-f_m) \tag{A.44}$$

Similarly, we can separate the "diathermal" contribution (A.34) as

$$M_{\mu\nu} = M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu} \tag{A.45}$$

where $M^{(1)}_{\mu\nu}$ and $M^{(2)}_{\mu\nu}$ refer to the first and second term in (A.34) respectively. Using the completeness relation $\sum_m |m\rangle\langle m| = 1$ it is easy to show that

$$M^{(2)}_{\mu\nu} = \frac{i}{4}\sum_{mn}(f_n-f_m)V_\mu^{nm}V_\nu^{mn}. \tag{A.46}$$

Comparison of (A.44) and (A.46) yields $M^{(2)}_{\mu\nu} + K^{(2)}_{\mu\nu} = 0$. Therefore the thermal response coefficient is given by

$$L^Q_{\mu\nu} = K^{(1)}_{\mu\nu} + M^{(1)}_{\mu\nu}. \tag{A.47}$$

Utilizing commutation relationships (2.33), $M^{(1)}_{\mu\nu}$ can be expressed in the form:

$$M^{(1)}_{\mu\nu} = \int \eta f(\eta) \mathrm{Tr}\left(\delta(\eta - \hat{H}_0)(x^\mu V^\nu - x^\nu V^\mu)\right) d\eta \tag{A.48}$$

where the integral extends over the entire real line. The rest of the section follows closely Smrčka and Středa [50]. We define the resolvents $G^\pm$:

$$G^\pm \equiv (\eta \pm i0 - \hat{H}_0)^{-1} \tag{A.49}$$

and operators

$$\begin{aligned} A(\eta) &= i\mathrm{Tr}\left(V_\mu \tfrac{dG^+}{d\eta} V_\nu \delta(\eta - \hat{H}_0) - V_\mu \delta(\eta - \hat{H}_0) V_\nu \tfrac{dG^-}{d\eta}\right) \\ B(\eta) &= i\mathrm{Tr}\left(V_\mu G^+ V_\nu \delta(\eta - \hat{H}_0) - V_\mu \delta(\eta - \hat{H}_0) V_\nu G^-\right) \end{aligned} \tag{A.50}$$

To facilitate Sommerfeld expansion we note that response coefficients $K^{(1)}_{\mu\nu}(T)$, $M^{(1)}_{\mu\nu}(T)$, $\sigma^s_{\mu\nu}(T)$ have generic form

$$L(T) = \int f(\eta) l(\eta) d\eta, \tag{A.51}$$

and after integration by parts

$$L(T) = -\int \frac{df}{d\eta} \tilde{L}(\eta) d\eta. \tag{A.52}$$

Here $\tilde{L}(\xi)$ is defined as

$$\tilde{L}(\xi) \equiv \int_{-\infty}^{\xi} l(\eta) d\eta. \tag{A.53}$$

Note that $L(T=0) = \tilde{L}(\xi=0)$ and in particular the spin conductivity at $T = 0$ satisfies $\sigma^s_{\mu\nu}(T=0) = \tilde{\sigma}^s_{\mu\nu}(\xi=0)$. Identities (A.52) and (A.53) will enable us to express the coefficients at finite temperature through the coefficients at zero temperature. For example, it follows from Eq. (A.48) that

$$\tilde{M}^{(1)}(\xi) = \int_{-\infty}^{\xi} \eta \mathrm{Tr}\left(\delta(\eta - \hat{H}_0)(x^\mu V^\nu - x^\nu V^\mu)\right) \tag{A.54}$$

and from Eqs.(A.21, A.50)

$$\tilde{\sigma}^s_{\mu\nu}(\xi) = \frac{1}{4}\int_{-\infty}^{\xi} A(\eta)d\eta \tag{A.55}$$

Similarly, coefficient $K^{(1)}_{\mu\nu}$ from (A.43) can be written as

$$K^{(1)}_{\mu\nu} = -i\int d\eta f(\eta)\eta \sum \delta(\eta-\epsilon_n)\epsilon_m \left(\frac{V^{nm}_\mu V^{mn}_\nu}{(\eta-\epsilon_m+i0)^2} - \frac{V^{mn}_\mu V^{nm}_\nu}{(\eta-\epsilon_m-i0)^2}\right) \tag{A.56}$$

so that

$$\tilde{K}^{(1)}_{\mu\nu}(\xi) \equiv -i\int_{-\infty}^{\xi} d\eta\, \eta \sum \delta(\eta-\epsilon_n)\epsilon_m \left(\frac{V^{nm}_\mu V^{mn}_\nu}{(\eta-\epsilon_m+i0)^2} - \frac{V^{mn}_\mu V^{nm}_\nu}{(\eta-\epsilon_m-i0)^2}\right) \tag{A.57}$$

Using definitions (A.50), $\tilde{K}^{(1)}(\xi)$ can be expressed as

$$\tilde{K}^{(1)}_{\mu\nu}(\xi) = \int_{-\infty}^{\xi} \eta^2 A(\eta)d\eta + \int_{-\infty}^{\xi} \eta B(\eta)d\eta. \tag{A.58}$$

After integration by parts one obtains

$$\tilde{K}^{(1)}_{\mu\nu}(\xi) = \xi^2\int_{-\infty}^{\xi} A(\eta)d\eta + \int_{-\infty}^{\xi}(\eta^2-\xi^2)\left(A(\eta) - \frac{1}{2}\frac{dB}{d\eta}\right)d\eta. \tag{A.59}$$

As shown in Ref. ([50]) the last term in this expression is exactly compensated by $\tilde{M}^{(1)}(\xi)$: This becomes evident after noting that definitions in Eq. (A.50) imply

$$A - \frac{1}{2}\frac{dB(\eta)}{d\eta} = \frac{1}{2}\mathrm{Tr}\left[\frac{d\delta(\eta-\hat{H}_0)}{d\eta}(x^\mu V^\nu - x^\nu V^\mu)\right]. \tag{A.60}$$

Substituting the last identity into the Eq.(A.59) and integrating the second term by parts we obtain

$$\tilde{K}^{(1)}_{\mu\nu}(\xi) = \xi^2\int_{-\infty}^{\xi} A(\eta)d\eta - \int_{-\infty}^{\xi}\eta\mathrm{Tr}\left(\delta(\eta-\hat{H}_0)(x^\mu V^\nu - x^\nu V^\mu)\right)d\eta \tag{A.61}$$

The second term here is equal to $-M^{(1)}_{\mu\nu}$ from (A.54). After the cancellation the result simply reads:

$$\tilde{L}^Q_{\mu\nu}(\xi) = \tilde{K}^{(1)}_{\mu\nu}(\xi) + \tilde{M}^{(1)}_{\mu\nu}(\xi) = \xi^2\int_{-\infty}^{\xi} A(\eta)d\eta. \tag{A.62}$$

Or, using (A.55)

$$\tilde{L}^{Q}_{\mu\nu}(\xi) = \left(\frac{2\xi}{\hbar}\right)^2 \tilde{\sigma}^{s}_{\mu\nu}(\xi). \tag{A.63}$$

Finally, from Eq.(A.52) we find

$$L^{Q}_{\mu\nu}(T) = -\frac{4}{\hbar^2}\int \frac{df(\eta)}{d\eta}\eta^2 \tilde{\sigma}^{s}_{\mu\nu}(\eta)d\eta. \tag{A.64}$$

Appendix B

Field theory of quasiparticles and vortices

B.1 Jacobian $\mathcal{L}_0$

Here we derive the explicit form of the "Jacobian" $\mathcal{L}_0$ for two cases of interest: *i*) the thermal vortex-antivortex fluctuations in 2D layers and *ii*) the spacetime vortex loop excitations relevant for low temperatures ($T \ll T^*$) deep in the underdoped regime.

B.1.1 2D thermal vortex-antivortex fluctuations

In order to perform specific computations we have to adopt a model for vortex-antivortex excitations. We will use a 2D Coulomb gas picture of vortex-antivortex plasma. In this model (anti)vortices are either point-like objects or are assumed to have a small hard-disk radius of size of the coherence length ξ_0 which emulates the core region. As long as $\xi_0 \ll n^{-\frac{1}{2}}$, where $n = n_v + n_a$ is the average density of vortex defects, the two models lead to very similar results and both undergo a vortex-antivortex pair unbinding transition of the Kosterlitz-Thouless variety.

Above the transition we have

$$\exp\left[-\beta \int d^2 r \mathcal{L}_0\right] = 2^{-N_l} \sum_{A,B} \int \mathcal{D}\varphi(\mathbf{r}) \tag{B.1}$$
$$\times \delta[\nabla \times \mathbf{v} - \tfrac{1}{2}\nabla \times (\nabla\varphi_A + \nabla\varphi_B)]$$
$$\times \delta[\nabla \times \mathbf{a} - \tfrac{1}{2}\nabla \times (\nabla\varphi_A - \nabla\varphi_B)] \quad .$$

The phase $\varphi(\mathbf{r})$ is due solely to vortices and we can rewrite (B.1) as:

$$\sum_{N_v,N_a} \frac{2^{-N_l}}{N_v! N_a!} \sum_{A,B} \prod_i^{N_v} \int d^2 r_i \prod_j^{N_a} \int d^2 r_j e^{-\beta E_c (N_v + N_a)} \tag{B.2}$$
$$\times \delta\big[\rho_v(\mathbf{r}) - \sum_i^{N_v} \delta(\mathbf{r} - \mathbf{r}_i)\big] \delta\big[\rho_a(\mathbf{r}) - \sum_j^{N_a} \delta(\mathbf{r} - \mathbf{r}_j)\big]$$
$$\times \delta\big[b(\mathbf{r}) - \pi \sum_i^{N_v^A} \delta(\mathbf{r} - \mathbf{r}_i^A) + \pi \sum_i^{N_v^B} \delta(\mathbf{r} - \mathbf{r}_i^B)$$
$$+ \pi \sum_j^{N_a^A} \delta(\mathbf{r} - \mathbf{r}_j^A) - \pi \sum_i^{N_a^B} \delta(\mathbf{r} - \mathbf{r}_j^B)\big] \quad .$$

Here $N_v(N_a)$ is the number of free vortices (antivortices), $N_l = N_v + N_a$, $\mathbf{r}_i$ ($\mathbf{r}_j$) are vortex (antivortex) coordinates and $\rho_v(\mathbf{r})$ ($\rho_a(\mathbf{r})$) are the corresponding densities. $b(\mathbf{r}) = (\nabla \times a(\mathbf{r}))_z = \pi(\rho_v^A - \rho_v^B - \rho_a^A + \rho_a^B)$ and E_c is the core energy which we have absorbed into $\mathcal{L}_0$ for convenience. We now express the above δ-functions as functional integrals over three new fields: $d_v(\mathbf{r})$, $d_a(\mathbf{r})$ and $\kappa(\mathbf{r})$:

$$\sum_{N_v,N_a} \frac{2^{-N_l}}{N_v! N_a!} \sum_{A,B} \prod_i^{N_v} \int d^2 r_i \prod_j^{N_a} \int d^2 r_j e^{-\beta E_c (N_v + N_a)} \tag{B.3}$$
$$\times \int \mathcal{D}d_v \mathcal{D}d_a \mathcal{D}\kappa \exp\big\{ i \int d^2 r d_v (\rho_v(\mathbf{r}) - \sum_i^{N_v} \delta(\mathbf{r} - \mathbf{r}_i))$$
$$+ i \int d^2 r d_a (\rho_a(\mathbf{r}) - \sum_j^{N_a} \delta(\mathbf{r} - \mathbf{r}_j))$$
$$+ i \int d^2 r \kappa \big[b(\mathbf{r}) - \pi \sum_i^{N_v^A} \delta(\mathbf{r} - \mathbf{r}_i^A) + \pi \sum_i^{N_v^B} \delta(\mathbf{r} - \mathbf{r}_i^B)$$
$$+ \pi \sum_j^{N_a^A} \delta(\mathbf{r} - \mathbf{r}_j^A) - \pi \sum_i^{N_a^B} \delta(\mathbf{r} - \mathbf{r}_j^B)\big]\big\} \quad .$$

The integration over δ-functions in the exponential is easily performed and the summation over $A(B)$ labels can be carried out explicitly to obtain:

$$\int \mathcal{D}d_v \mathcal{D}d_a \mathcal{D}\kappa \exp\left[i \int d^2r \big(d_v\rho_v + d_a\rho_a + \kappa b\big)\right] \tag{B.4}$$
$$\times \sum_{N_v,N_a} \frac{e^{-\beta E_c N_v}}{N_v!} \prod_i^{N_v} \int d^2r_i \exp(-i d_v(\mathbf{r}_i)) \cos(\pi\kappa(\mathbf{r}_i))$$
$$\times \frac{e^{-\beta E_c N_a}}{N_a!} \prod_j^{N_a} \int d^2r_j \exp(-i d_a(\mathbf{r}_j)) \cos(\pi\kappa(\mathbf{r}_j)).$$

In the thermodynamic limit the sum (B.4) is dominated by $N_{v(a)} = \langle N_{v(a)}\rangle$, where $\langle N_{v(a)}\rangle$ is the average number of free (anti)vortices determined by solving the full problem. Furthermore, as $\langle N_{v(a)}\rangle \to \infty$ in the thermodynamic limit the integration over $d_v(\mathbf{r})$, $d_a(\mathbf{r})$ and $\kappa(\mathbf{r})$ can be performed in the saddle-point approximation leading to the following saddle-point equations:

$$-\rho_v(\mathbf{r}) + \langle N_v\rangle \Omega_v(\mathbf{r}) = 0 \tag{B.5}$$

$$-\rho_a(\mathbf{r}) + \langle N_a\rangle \Omega_a(\mathbf{r}) = 0 \tag{B.6}$$

$$-b(\mathbf{r}) + [\langle N_v\rangle \Omega_v(\mathbf{r}) + \langle N_a\rangle \Omega_a(\mathbf{r})] \tag{B.7}$$
$$\times \pi \tanh(\pi\kappa(\mathbf{r})) = 0$$

with

$$\Omega_m(\mathbf{r}) = \frac{e^{d_m(\mathbf{r})} \cosh(\pi\kappa(\mathbf{r}))}{\int d^2r' e^{d_m(\mathbf{r}')} \cosh(\pi\kappa(\mathbf{r}'))}, \qquad m = a, v.$$

Eqs. (B.5-B.7) follow from functional derivatives of (B.4) with respect to $d_v(\mathbf{r})$, $d_a(\mathbf{r})$ and $\kappa(\mathbf{r})$, respectively. We have also built in the fact that the saddle-point solutions occur at $d_v \to i d_v$, $d_a \to i d_a$, $\kappa \to i\kappa$.

The saddle-point equations (B.5-B.7) can be solved exactly leading to:

$$d_v(\mathbf{r}) = \ln \rho_v(\mathbf{r}) - \ln\cosh(\pi\kappa(\mathbf{r})) \tag{B.8}$$

$$d_a(\mathbf{r}) = \ln \rho_a(\mathbf{r}) - \ln\cosh(\pi\kappa(\mathbf{r})) \tag{B.9}$$

where

$$\kappa(\mathbf{r}) = \frac{1}{\pi} \tanh^{-1}\left[\frac{b(\mathbf{r})}{\pi(\rho_v(\mathbf{r}) + \rho_a(\mathbf{r}))}\right]. \tag{B.10}$$

Inserting (B.8-B.10) back into (B.4) finally gives the entropic part of $\mathcal{L}_0/T$:

$$\begin{aligned}\rho_v &\quad \ln\rho_v + \rho_a \ln\rho_a - \frac{1}{\pi}(\nabla\times\mathbf{a})_z \tanh^{-1}\left[\frac{(\nabla\times\mathbf{a})_z}{\pi(\rho_v+\rho_a)}\right] \\ + &\quad (\rho_v+\rho_a)\ln\cosh\tanh^{-1}\left[\frac{(\nabla\times\mathbf{a})_z}{\pi(\rho_v+\rho_a)}\right], \end{aligned} \tag{B.11}$$

where $\rho_{v(a)}(\mathbf{r})$ are densities of *free* (anti)vortices. We display $\mathcal{L}_0$ in this form to make contact with familiar physics: the first two terms in (B.11) are the entropic contribution of free (anti)vortices and the Doppler gauge field $\nabla\times\mathbf{v}\to\pi(\rho_v-\rho_a)$ ($\langle\nabla\times\mathbf{v}\rangle = 0$). The last two terms encode the "Berry phase" physics of topological frustration. Note that the "Berry" magnetic field $b = (\nabla\times\mathbf{a})_z$ couples directly only to the *total density* of vortex defects $\rho_v+\rho_a$ and is insensitive to the vortex charge. This is a reflection of the Z_2 symmetry of the original problem defined on discrete (i.e., not coarse-grained) vortices. We write $\rho_{v(a)}(\mathbf{r}) = \langle\rho_{v(a)}\rangle + \delta\rho_{v(a)}(\mathbf{r})$ and expand (B.11) to leading order in $\delta\rho_{v(a)}$ and $\nabla\times\mathbf{a}$:

$$\mathcal{L}_0/T \to (\nabla\times\mathbf{v})^2/(2\pi^2 n_l) + (\nabla\times\mathbf{a})^2/(2\pi^2 n_l), \tag{B.12}$$

where $n_l = \langle\rho_v\rangle + \langle\rho_a\rangle$ is the average density of free vortex defects. Both $\mathbf{v}$ and $\mathbf{a}$ have a Maxwellian *bare* stiffness and are *massless* in the normal state. As one approaches T_c, $n_l \sim \xi_{\rm sc}^{-2} \to 0$, where $\xi_{\rm sc}(x,T)$ is the superconducting correlation length, and $\mathbf{v}$ and $\mathbf{a}$ become *massive* (see the main text).

B.1.2 Quantum fluctuations of (2+1)D vortex loops

The expression for $\mathcal{L}_0[j_\mu]$ given by Eq. (3.39) follows directly once the system contains unbound vortex loops in its ground state and thus can respond to the external perturbation $A_\mu^{\rm ext}$ over arbitrary large distances in (2+1)-dimensional spacetime. This is already clear at intuitive level if we just think of the geometry of infinite versus finite loops and the fact that only unbound loops allow vorticity fluctuations, described by $\langle j_\mu(q) j_\nu(-q)\rangle$, to proceed unhindered. Still, it is useful to derive (3.39) and its consequences belabored in Section II within an explicit model for vortex loop fluctuations.

Here we consider fluctuating vortex loops in continuous (2+1)D spacetime and compute $\mathcal{L}_0[j_\mu]$ using duality map to the relativistic Bose superfluid [86]. We again start by using our model of a large gap s-wave superconductor which enables us to neglect a_μ in the fermion action and integrate over it in the expression for $\mathcal{L}_0[v_\mu, a_\mu]$ (3.11). This gives:

$$e^{-\int d^3x \mathcal{L}_0} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{l=1}^{N} \oint \mathcal{D}x_l[s_l] \delta\big(j_\mu(x) - n_\mu(x)\big) e^{-\tilde{S}} \tag{B.13}$$

where

$$\tilde{S} = \sum_{l=1}^{N} S_0 \oint ds_l + \frac{1}{2} \sum_{l,l'=1}^{N} \oint ds_l \oint ds_{l'} g(|x_l[s_l] - x_{l'}[s_{l'}]|) \tag{B.14}$$

and

$$\big(\partial \times \partial\varphi(x)\big)_\mu = 2\pi n_\mu(x) = 2\pi \sum_{l}^{N} \oint_L dx_{l\mu} \delta(x - x_l[s_l]) \ . \tag{B.15}$$

In the above equations N is the number of loops, s_l is the Schwinger proper time (or "proper length") of loop l, S_0 is the action per unit length associated with motion of vortex cores in (2+1)-dimensional spacetime (in an analogous 3D model this would be ε_c/T, where ε_c is the core line energy), $g(|x_l[s_l] - x_{l'}[s_{l'}]|)$ is the short range penalty for core overlap and L denotes a line integral. We kept our practice of including core terms independent of vorticity into $\mathcal{L}_0$. Note that vortex loops must be periodic along τ reflecting the original periodicity of $\varphi(\mathbf{r}, \tau)$.

We can think of vortex loops as worldline trajectories of some relativistic charged (complex) bosons (charged since the loops have two orientations) in two spatial dimensions. $\mathcal{L}_0$ without the δ-function describes the vacuum Lagrangian of such a theory, with vortex loops representing particle-antiparticle virtual creation and annihilation process. The duality map is based on the following relation between the Green function of free charged bosons and a gas of free oriented loops:

$$G(x) = \langle \phi(0)\phi^*(x) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot x}}{k^2 + m_d^2} = \int_0^\infty ds e^{-sm_d^2}$$

$$\times \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x - sk^2} = \int_0^\infty ds e^{-sm_d^2} \left(\frac{1}{4\pi s}\right)^{\frac{3}{2}} e^{-\frac{x^2}{4s}}, \tag{B.16}$$

where m_d is the mass of the complex dual field $\phi(x)$. Within the Feynman path integral representation we can write:

$$\left(\frac{1}{4\pi s}\right)^{3/2} e^{-\frac{1}{4}x^2/s} = \int_{x(0)=0}^{x(s)=x} \mathcal{D}x(s') \exp\left[-\frac{1}{4}\int_0^s ds' \dot{x}^2(s')\right] \ , \tag{B.17}$$

where $\dot{x} \equiv dx/ds$. Furthermore, by simple integration (B.16) can be manipulated into

$$\begin{aligned}\mathrm{Tr}\left[\ln(-\partial^2 + m_d^2)\right] &= -\int_0^\infty \frac{ds}{s} e^{-sm_d^2} \int \frac{d^3k}{(2\pi)^3} e^{-sk^2} \\ &= -\int_0^\infty \frac{ds}{s} e^{-sm_d^2} \oint \mathcal{D}x(s') \exp\left[-\frac{1}{4}\int_0^s ds' \dot{x}^2(s')\right]\end{aligned} \tag{B.18}$$

where the path integral now runs over closed loops ($x(s) = x(0)$). In the dual theory this can be reexpressed as

$$\mathrm{Tr}\left[\ln(-\partial^2 + m_d^2)\right] = \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + m_d^2) \ . \tag{B.19}$$

Combining (B.18) and (B.19) and using

$$\mathrm{Tr}\left[\ln(-\partial^2 + m_d^2)\right] = \ln \mathrm{Det}(-\partial^2 + m_d^2) \equiv \mathcal{W} \tag{B.20}$$

finally leads to:

$$e^{-\mathcal{W}} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{l=1}^{N} \left[\int_0^\infty \frac{ds_l}{s_l} e^{-s_l m_d^2} \oint \mathcal{D}x(s_l')\right] \times \exp\left[-\frac{1}{4}\sum_{l=1}^{N} \int_0^{s_l} ds_l' \dot{x}^2(s_l')\right] \ . \tag{B.21}$$

This is nothing else but the partition function of the free loop gas. The size of loops is regulated by m_d. As $m_d \to 0$ the average loop size diverges. On the other hand, through $\mathcal{W}$ (B.20), we can also think of (B.21) as the partition function of the free bosonic theory.

To exploit this equivalence further we write

$$Z_d = \int \mathcal{D}\phi^* \mathcal{D}\phi e^{-\int d^3x \mathcal{L}_d} \ , \tag{B.22}$$

where

$$\mathcal{L}_d = |\partial\phi|^2 + m_d^2|\phi|^2 + \frac{g}{2}|\phi|^4 \ , \tag{B.23}$$

and argue that Z_d describes vortex loops with short range interactions in (B.13). This can be easily demonstrated by decoupling $|\phi|^4$ through Hubbard-Stratonovich transformation and retracing the above steps. The reader is referred to the book by Kleinert for further details of the above duality mapping [86].

We can now rewrite the δ-function in (B.13) as

$$\delta\big(j_\mu(x) - n_\mu(x)\big) \to \int \mathcal{D}\kappa_\mu \exp\big(i \int d^3x \kappa_\mu (j_\mu - n_\mu)\big) \tag{B.24}$$

and observe that in the above language of Feynman path integrals in proper time $i\int d^3x\kappa_\mu n_\mu$ (B.15) assumes the meaning of a particle current three-vector n_μ coupled to a three-vector potential κ_μ. Employing the same arguments that led to (B.23) we now have:

$$\mathcal{L}_d[\kappa_\mu] = |(\partial - i\kappa)\phi|^2 + m_d^2|\phi|^2 + \frac{g}{2}|\phi|^4 \quad , \tag{B.25}$$

which leads to

$$e^{-\int d^3x \mathcal{L}_0[j_\mu]} \longrightarrow \int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}\kappa_\mu e^{-\int d^3x(-i\kappa_\mu j_\mu + \mathcal{L}_d[\kappa_\mu])} \quad . \tag{B.26}$$

In the pseudogap state vortex loop unbinding causes loss of superconducting phase coherence. In the dual language, the appearance of such infinite loops as $m_d \to 0$ implies superfluidity in the system of charged bosons described by $\mathcal{L}_d$ (B.23). The dual off-diagonal long range order (ODLRO) in $\langle\phi(x)\phi^*(x')\rangle$ means that there are worldline trajectories that connect points x and x' over infinite spacetime distances – these infinite worldlines are nothing else but unbound vortex paths in this "vortex loop condensate". Thus, the dual and the true superconducting ODLRO are two opposite sides of the same coin.

$\mathcal{L}_d[\kappa_\mu]$ is the Lagrangian of this dual superfluid in presence of the external vector potential κ_μ. In the ordered phase, the response of the system is just the dual version of the Meissner effect. Consequently, upon functional integration over ϕ we are allowed to write:

$$e^{-\int d^3x \mathcal{L}_0[j_\mu]} = \int \mathcal{D}\kappa_\mu e^{-\int d^3x\big(-i\kappa_\mu j_\mu + M_d^2\kappa_\mu\kappa_\mu\big)} \quad , \tag{B.27}$$

where $M_d^2 = |\langle\phi\rangle|^2$, with $\langle\phi\rangle$ being the dual order parameter. The remaining functional integration over κ_μ finally results in:

$$\mathcal{L}_0[j_\mu] \to \frac{j_\mu j_\mu}{4|\langle\phi\rangle|^2} \quad . \tag{B.28}$$

We have tacitly assumed that the system of loops is isotropic. The intrinsic anisotropy of the (2+1)D theory is easily reinstated and (B.28) becomes Eq. (3.39) of the main text.

It is now time to recall that we are interested in a d-wave superconductor. This means we must restore a_μ to the problem. To accomplish this we engage in a bit of thievery: imagine now that it was v_μ whose coupling to fermions was negligible and we could integrate over it in (3.11). We would then be left with only the δ-function containing a_μ. Actually, we can compute such $\mathcal{L}_0[b_\mu/\pi]$, where $\partial \times \partial a = b$, without any additional work. Note that $\mathcal{L}_0$ contains only vorticity independent terms. We can equally well proclaim that it is the $A(B)$ labels that determine the true orientation of our loops while the actual vorticity is simply a gauge label – in essence, v_μ and a_μ trade places. After the same algebra as before we obtain:

$$\mathcal{L}_0[b_\mu/\pi] \to \frac{b_\mu b_\mu}{4\pi^2|\langle\phi\rangle|^2} \quad , \tag{B.29}$$

which is just the Maxwell action for a_μ.

Of course, this simple argument that led to (B.29) is illegal. We cannot just forget v_μ. If we did we would have no right to coarse-grain a_μ to begin with and would have to face up to its purely Z_2 character (see Section II). Still, the above reasoning does illustrate that the Maxwellian stiffness of a_μ follows the same pattern as that of v_μ: both are determined by the order parameter $\langle\phi\rangle$ of condensed *dual* superfluid. Thus, we can write the correct form of $\mathcal{L}_0$, with both v_μ and a_μ fully included into our accounting, as

$$\mathcal{L}_0[v_\mu, a_\mu] \to \frac{(\partial \times v)_\mu (\partial \times v)_\mu}{4\pi^2|\langle\phi\rangle|^2} + \frac{(\partial \times a)_\mu (\partial \times a)_\mu}{4\pi^2|\langle\phi\rangle|^2} \quad , \tag{B.30}$$

where our ignorance is now stored in computing the actual value of $\langle\phi\rangle$ from the original parameters of the d-wave superconductor problem. With anisotropy restored this is precisely Eq. (3.45) of Section II.

B.2 Feynman integrals in QED$_3$

Many of the integrals encountered here are considered standard in particle physics. Since the techniques involved are not as common in the condensed matter physics we provide some of the technical details in this Appendix. A more in-depth discussion can be found in many field theory textbooks [76].

B.2.1 Vacuum polarization bubble

The vacuum polarization Eq. (3.59) can be written more explicitly as

$$\Pi_{\mu\nu}(q) = 2N\mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu]I_{\alpha\beta}(q) \tag{B.31}$$

with

$$I_{\alpha\beta}(q) = \int \frac{d^3k}{(2\pi)^3}\frac{k_\alpha(k_\beta+q_\beta)}{k^2(k+q)^2}. \tag{B.32}$$

The integrals of this type are most easily evaluated by employing the Feynman parametrization [76]. This consists in combining the denominators using the formula

$$\frac{1}{A^aB^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\frac{x^{a-1}(1-x)^{b-1}}{[xA+(1-x)B]^{a+b}}, \tag{B.33}$$

valid for any positive real numbers a, b, A, B. Setting $A = k^2$ and $B = (k+q)^2$ allows us to rewrite (B.32) as

$$I_{\alpha\beta}(q) = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{k_\alpha(k_\beta+q_\beta)}{[(k+(1-x)q)^2+x(1-x)q^2]^2}. \tag{B.34}$$

We now shift the integration variable $k \to k-(1-x)q$ to obtain

$$I_{\alpha\beta}(q) = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{k_\alpha k_\beta - x(1-x)q_\alpha q_\beta}{[k^2+x(1-x)q^2]^2}, \tag{B.35}$$

where we have dropped terms odd in k which vanish by symmetry upon integration. We now notice that $k_\alpha k_\beta$ term will only contribute if $\alpha = \beta$ and we can therefore replace it in Eq. (B.35) by $\frac{1}{3}\delta_{\alpha\beta}k^2$. With this replacement the angular integrals are trivial and we have

$$I_{\alpha\beta}(q) = \frac{1}{2\pi^2}\int_0^1 dx \int_0^\infty dk k^2\frac{\frac{1}{3}\delta_{\alpha\beta}k^2 - x(1-x)q_\alpha q_\beta}{[k^2+x(1-x)q^2]^2}. \tag{B.36}$$

The only remaining difficulty stems from the fact that the integral formally diverges at the upper bound. This divergence is an artifact of our linearization of the fermionic spectrum which breaks down for energies approaching the superconducting gap value. It is therefore completely legitimate to introduce an ultraviolet cutoff. Such UV cutoffs however tend to interfere with gauge invariance preservation of which is crucial in this computation. A more physical way of regularizing the integral Eq. (B.36) is to recall that the gauge field must remain massless, i.e. $\Pi_{\mu\nu}(q \to 0) = 0$. To enforce this property we write Eq. (B.31) as

$$\Pi_{\mu\nu}(q) = 2N\mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu][I_{\alpha\beta}(q) - I_{\alpha\beta}(0)], \tag{B.37}$$

and we see that proper regularization of Eq. (B.36) involves subtracting the value of the integral at $q = 0$. The remaining integral is convergent and elementary; explicit evaluation gives

$$I_{\alpha\beta}(q) - I_{\alpha\beta}(0) = -\frac{|q|}{64}\left(\delta_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\right). \tag{B.38}$$

Inserting this in (B.37) and working out the trace using Eqs. (3.54) and (3.55) we find the result (3.60). Identical result can be obtained using dimensional regularization.

B.2.2 TF self energy: Lorentz gauge

For simplicity we evaluate the self energy (3.64) in the Lorentz gauge ($\xi = 0$). Extension to arbitrary covariant gauge is trivial. Eq. (3.64) can be written as

$$\Sigma_L(k) = -\frac{2}{\pi^3 N}\gamma_\mu I_\mu(k) \tag{B.39}$$

with

$$I_\mu(k) = \int d^3q q_\mu \frac{q\cdot(k+q)}{|q|^3(k+q)^2}, \tag{B.40}$$

where we used an identity

$$\gamma_\mu\gamma_\alpha\gamma_\nu\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) = -2\not{q}\frac{q_\alpha}{q^2}.$$

Since the only 3-vector available is k, clearly the vector integral $I_\mu(k)$ can only have components in the k_μ direction, $I_\mu(k) = \mathcal{C}(k)k_\mu/k^2$. By forming a scalar product

$k_\mu I_\mu(k)$ we obtain

$$\mathcal{C}(k) = k_\mu I_\mu(k) = \int d^3q \frac{(q\cdot k)(q\cdot k + q^2)}{|q|^3(k+q)^2}. \tag{B.41}$$

Combining the denominators using Eq. (B.33) and following the same steps as in the computation of polarization bubble above we obtain

$$\begin{aligned}\mathcal{C}(k) &= \frac{3}{2}\int_0^1 dx\sqrt{x}\int d^3q \\ &\times \frac{(2x-1)(k\cdot q)^2 - (1-x)k^2q^2 + x(1-x)^2k^4}{[q^2+x(1-x)k^2]^{5/2}}.\end{aligned} \tag{B.42}$$

The angular integrals are trivial and after introducing an ultraviolet cutoff Λ and rescaling the integration variable by $|k|$ the integral becomes

$$\mathcal{C}(k) = 2\pi k^2 \int_0^1 dx\sqrt{x}\int_0^{\frac{\Lambda}{|k|}} dq \frac{(5x-4)q^4 + 3x(1-x)^2q^2}{[q^2+x(1-x)]^{5/2}}. \tag{B.43}$$

The remaining integrals are elementary. Isolating the leading infrared divergent term we obtain

$$\mathcal{C}(k) \to -\frac{4\pi}{3}k^2 \ln\frac{\Lambda}{|k|}. \tag{B.44}$$

Substituting this to Eq. (B.39) we get the self energy (3.65).

B.3 Feynman integrals for anisotropic case

B.3.1 Self energy

As shown in the Section V, to first order in $1/N$ expansion, the topological fermion self energy can be reduced to

$$\Sigma_n(q) = \int \frac{d^3k}{(2\pi)^3}\frac{(q-k)_\lambda(2g^n_{\lambda\mu}\gamma^n_\nu - \gamma^n_\lambda g^n_{\mu\nu})D_{\mu\nu}(k)}{(q-k)_\mu g^n_{\mu\nu}(q-k)_\nu}. \tag{B.45}$$

where $D_{\mu\nu}(q)$ is the screened gauge field propagator evaluated in the Section 3. In order to perform the radial integral, we rescale the momenta as

$$K_\mu = \sqrt{g^n}_{\mu\nu}k_\nu; \quad Q_\mu = \sqrt{g^n}_{\mu\nu}q_\nu \tag{B.46}$$

and obtain

$$\Sigma_n(q)=\int\frac{d^3K}{(2\pi)^3}\frac{(Q-K)_\lambda(2\sqrt{g^n}_{\lambda\mu}\gamma^n_\nu-\gamma_\lambda g^n_{\mu\nu})D_{\mu\nu}\left(\frac{K_\nu}{\sqrt{g^n}_{\mu\nu}}\right)}{v_F v_\Delta(Q-K)^2}. \tag{B.47}$$

At low energies we can neglect the contribution from the bare field stiffness ρ in the gauge propagator (see Section 3) and the resulting $D_{\mu\nu}(q)$ exhibits $\frac{1}{q}$ scaling

$$D_{\mu\nu}\left(\frac{K_\nu}{\sqrt{g^n}_{\mu\nu}}\right)=\frac{F_{\mu\nu}(\theta,\phi)}{|K|} \tag{B.48}$$

Now we can explicitly integrate over the magnitude of the rescaled momentum K by introducing an upper cut-off Λ and in the leading order we find that

$$\int_0^\Lambda dK\frac{K^2(Q-K)_\lambda}{K(Q-K)^2}=-\Lambda\hat{K}_\lambda+\ln\left(\frac{\Lambda}{Q}\right)\left(Q_\lambda-2\hat{K}_\lambda\hat{K}\cdot Q\right) \tag{B.49}$$

where $\hat{K}=(\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$. Since $\hat{K}_\mu$ is odd under inversion while $F_{\mu\nu}$ is even, it is not difficult to see that the term proportional to Λ vanishes upon the angular integration. Thus

$$\Sigma_n(q)=\int\frac{d\Omega}{(2\pi)^3 v_F v_\Delta}\left(Q_\lambda-2\hat{K}_\lambda\hat{K}\cdot Q\right)(\times)$$
$$(\times)(2\sqrt{g^n}_{\lambda\mu}\gamma^n_\nu-\gamma_\lambda g^n_{\mu\nu})F_{\mu\nu}(\theta,\phi)\ln\left(\frac{\Lambda}{Q}\right). \tag{B.50}$$

Using the fact that the diagonal elements of $F_{\mu\nu}(\theta,\phi)$ are even under parity, while the off-diagonal elements are odd under parity, the above expression can be further simplified to

$$\Sigma_n(q)=-\sum_\mu(\gamma^n_\mu q_\mu\eta^n_\mu)\ln\left(\frac{\Lambda}{\sqrt{q_\alpha g^n_{\alpha\beta}q_\beta}}\right) \tag{B.51}$$

where the coefficients η^n_μ are pure numbers depending on the bare anisotropy and are thereby reduced to a quadrature

$$\eta^n_\mu=\int\frac{d\Omega}{v_F v_\Delta(2\pi)^3}\Bigg((2\hat{K}_\mu\hat{K}_\mu-1)(2g^n_{\mu\mu}F_{\mu\mu}-\sum_\nu g^n_{\nu\nu}F_{\nu\nu})$$
$$+\sum_{\nu\neq\mu}4\hat{K}_\mu\hat{K}_\nu\sqrt{g^n}_{\mu\mu}\sqrt{g^n}_{\nu\nu}F_{\mu\nu}\Bigg). \tag{B.52}$$

Repeated indices are not summed in the above expression, unless explicitly indicated. In the case of weak anisotropy ($v_F = 1 + \epsilon, v_\Delta = 1$) we can show that the Eq.(B.52) reduces to Eqs.(3.92-3.94).

Consider now the effect of the (covariant) gauge fixing term on η_μ. Let us define the part of $F_{\mu\nu}$ which depends on the gauge fixing parameter ξ as $F^{(\xi)}_{\mu\nu}$. The general form of this term is $F^{(\xi)}_{\mu\nu} = \xi\ k_\mu k_\nu f(k)$ where $f(k)$ is a scalar function of all three components of k_μ; $f(k)$ does in general depend on the anisotropy. Upon rescaling with the nodal metric, see Eq.(B.46), we have $F^{(\xi)}_{\mu\nu} = \xi \frac{1}{\sqrt{g_{\mu\mu}}} K_\mu \frac{1}{\sqrt{g_{\nu\nu}}} K_\nu \tilde{f}(K)$, where $\tilde{f}(K)$ is the corresponding scalar function of K_μ. Substituting $F^{(\xi)}_{\mu\nu}$ into the Eq. (B.52) we find

$$\eta^\xi_\mu = \xi \int \frac{d\Omega\ \tilde{f}(K)}{v_F v_\Delta (2\pi)^3} \Big((2\hat{K}_\mu\hat{K}_\mu - 1)(2\hat{K}_\mu\hat{K}_\mu - \sum_\nu \hat{K}_\nu\hat{K}_\nu) + \sum_{\nu\neq\mu} 4\hat{K}_\mu\hat{K}_\nu\hat{K}_\mu\hat{K}_\nu \Big), \quad \text{(B.53)}$$

where η^ξ_μ is the part of η_μ which comes entirely from the gauge fixing term. Using the fact that $\sum_\nu \hat{K}_\nu\hat{K}_\nu = 1$, it is a matter of simple algebra to show that

$$\eta^\xi_\mu = \xi \int \frac{d\Omega\ \tilde{f}(K)}{v_F v_\Delta (2\pi)^3} \quad \text{(B.54)}$$

i.e., the dependence on the index μ drops out! That means that the renormalization of η_μ due to the unphysical longitudinal modes is exactly the same for all of its components. Therefore, the difference in η_1 and η_2, which is related to the RG flow of the Dirac anisotropy comes entirely from the physical modes and is a gauge independent quantity. Note that this statement does not depend on the choice of the covariant gauge, i.e. on the exact form of the function f, only on the fact that the gauge is covariant.

B.4 Physical modes of a finite T QED$_3$

Free massless Abelian gauge theory in $d+1-$ space-time dimensions has $d-1$ physical degrees of freedom. The familiar example is the world we live in: light has only two (transverse) physical degrees of freedom. Using the modern terminology, photon is a spin 1 massless gauge particle, which restricts its spin projections to

$S = \pm 1$. These two degrees of freedom contribute to the specific heat of a photon gas.

When the gauge field interacts with matter, the situation is a little more involved. In $3+1D$ QED, matter effectively decouples from light and so the leading order contribution to the thermodynamics is just a sum of the contributions from the free fermions and free light. However, exchange of a massless particle introduces long range interactions between the fermions, thereby affecting the spectrum of the quantum theory. While there are no collective modes in a free theory, the coupled theory has multi-particle collective states in the spectrum, and these states will contribute to the thermodynamics of the coupled system.

One way of computing such contributions, is to integrate out the fermions. The partition function can be written as $Z = Z^0_{fermi} Z^{eff}_{gauge}$, where Z^0_{fermi} gives the free fermion contribution to the free energy, while Z^{eff}_{gauge} gives the contribution from the dressed gauge field. If this gauge field action is expanded to quadratic order, then the contribution from the gauge field can be computed exactly. It can be understood as fermion dressing or renormalization of the free gauge field. In general, there are two distinct processes: (1) the renormalization of the transverse components that contributed to thermodynamics even without any fermions, and (2) the renormalization of the longitudinal components, which did not contribute to thermodynamics in the absence of matter. The latter can be understood as arising solely from the collective modes of the fermions.

To see how all of this comes about in detail, consider the free gauge theory in Euclidean space-time:

$$\mathcal{L}_0(a_\mu) = \frac{1}{2e^2} a_\mu \Pi_{\mu\nu} a_\nu; \quad \Pi_{\mu\nu} = (k^2 \delta_{\mu\nu} - k_\mu k_\nu). \tag{B.55}$$

For future benefit, we shall rewrite the polarization tensor using

$$\begin{aligned} A_{\mu\nu} &= \left(\delta_{\mu 0} - \frac{k_\mu k_0}{k^2}\right) \frac{k^2}{\mathbf{k}^2} \left(\delta_{0\nu} - \frac{k_0 k_\nu}{k^2}\right) , \\ B_{\mu\nu} &= \delta_{\mu i} \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) \delta_{j\nu} , \end{aligned} \tag{B.56}$$

where $k_\mu = (\omega_n, \mathbf{k})$ and $\omega_n = 2\pi n T$ is the bosonic Matsubara frequency. It is straightforward to see that $\delta_{\mu\nu} - k_\mu k_\nu / k^2 = A_{\mu\nu} + B_{\mu\nu}$, and that the gauge field action is now

$$\mathcal{L}_0(a_\mu) = \frac{1}{2}\Pi^0_A a_\mu A_{\mu\nu} a_\nu + \frac{1}{2}\Pi^0_B a_\mu B_{\mu\nu} a_\nu \quad , \tag{B.57}$$

where $\Pi^0_A = \Pi^0_B = (\omega_n^2 + \mathbf{k}^2)/e^2$. We shall see later that to order $1/N$ the matter fields modify Π_A and Π_B, but not the generic form of Eq. (B.57).

Finite temperature breaks Lorentz invariance in that it introduces a fixed length in the temporal direction. Therefore, it is not convenient to use covariant gauge fixing (as at $T = 0$); instead we shall work in Coulomb's gauge $\nabla \cdot \mathbf{A} = 0$. This gauge eliminates one of the $d + 1$ degrees of freedom. In the free gauge theory, there is one more non-physical degree of freedom. To see that this is indeed so, integrate out the time component of the gauge field a_0. Schematically, this proceeds as follows:

$$\begin{aligned} \mathcal{L} &= \Pi_{00} a_0^2 + 2 a_i \Pi_{i0} a_0 + a_i \Pi_{ij} a_j; \quad i \neq 0; & \text{(B.58)} \\ &= \Pi_{00}\left(a_0 + \frac{1}{\Pi_{00}}\,\Pi_{0i} a_i\right)^2 - \frac{1}{\Pi_{00}}(\Pi_{i0} a_i)^2 + a_i \Pi_{ij} a_j & \text{(B.59)} \\ &= \frac{\mathbf{k}^2}{\omega_n^2 + \mathbf{k}^2}\Pi_A \tilde{a}_0^2 + \Pi_B\; a_i B_{ij} a_j, & \text{(B.60)} \end{aligned}$$

where we shifted the time component of the gauge field in the functional integral, $\tilde{a} = a_0 + \frac{1}{\Pi_{00}}\,\Pi_{0i} a_i$, and used the Eq. (B.56).

In the free field theory $\Pi_A = \Pi^0_A = (\omega_n^2 + \mathbf{k}^2)$ and $\Pi_B = \Pi^0_B = (\omega_n^2 + \mathbf{k}^2)$. Thus,

$$\mathcal{L}^0 = \mathbf{k}^2 \tilde{a}_0^2 + (\omega_n^2 + \mathbf{k}^2)\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) a_i a_j. \tag{B.61}$$

Since the coefficient in front of $\tilde{a}_0^2$ is independent of ω_n and therefore independent of temperature, this mode does not contribute to thermodynamics. In addition, $\delta_{ij} - k_i k_j/\mathbf{k}^2$ projects out the spatial longitudinal mode, and only $d - 1$ transverse massless modes contribute to thermodynamics.

When the gauge field interacts with the fermions, the effective quadratic action for the gauge field is modified by the fermion renormalization of the polarization functions: $\Pi_{A,B} = \Pi^0_{A,B} + \Pi^F_{A,B}$. While it is still true that the longitudinal spatial component of a_μ is projected out, it is no longer true that the coefficient in front of $\tilde{a}_0^2$ is independent of ω_n. We must therefore include this "mode" in computing the thermodynamical free energy. For instance, in the problem of electrodynamics of metal with a Fermi surface, this mode is sharp at the plasma frequency. In QED_3, inclusion of this mode is essential to preserve the "hyperscaling" of the $(z = 1)$ theory, in that $c_v \sim T^2$.

B.5 Finite Temperature Vacuum Polarization

In this section we shall compute the one loop polarization matrix of the finite temperature QED_3.

$$\Pi_{\mu\nu}(q) = \frac{N}{\beta}\sum_{\omega_n}\int \frac{d^2k}{(2\pi)^2} Tr[G_0(\omega_n, \mathbf{k})\gamma_\mu G_0(\Omega + \omega_n, \mathbf{k} + \mathbf{q})\gamma_\nu], \tag{B.62}$$

$$\Pi_{\mu\nu}(q) = \frac{N}{\beta}\sum_{\omega_n}\int \frac{d^2k}{(2\pi)^2}\frac{Tr[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta]k_\alpha(k_\beta + q_\beta)}{k^2(k+q)^2}, \tag{B.63}$$

where $k = (\omega_n, \mathbf{k}) = ((2n+1)\pi T, \mathbf{k})$. In the Euclidean space we have

$$Tr[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta] = 4(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) \tag{B.64}$$

and so

$$\Pi_{\mu\nu}(q) = \frac{4N}{\beta}\sum_{\omega_n}\int \frac{d^2k}{(2\pi)^2}\frac{L_{\mu\nu}(k;q)}{k^2(k+q)^2}, \tag{B.65}$$

where

$$L_{\mu\nu}(k;q) = k_\mu(k_\nu + q_\nu) + k_\nu(k_\mu + q_\mu) - \delta_{\mu\nu}k_\alpha(k_\alpha + q_\alpha) \tag{B.66}$$

where repeated indices (α) are summed.

At $T = 0$, the natural way to proceed is to use Feynman parameters. However, there is no advantage in doing so at finite T. Therefore, we shall first sum over the fermionic Matsubara frequencies ω_n; then, we perform the integral over $\mathbf{k}$ which we split into an integral over the angle and the integral over the magnitude of $\mathbf{k}$. The integral over the angle can be performed analytically. However, the last integral over the magnitude of $\mathbf{k}$ cannot be performed in the closed form. This happens quite frequently in theories dealing with degenerate fermions. However, our results will differ from such theories in an important *qualitative* way: there is no lengthscale associated with the Fermi energy and this prohibits the use of Sommerfeld expansion. Instead, the theory is quantum critical and the only scales are given by the external frequency and momentum, and by the temperature. We shall therefore use the expression for the polarization matrix $\Pi_{\mu\nu}$ without performing the final integral for general (Ω, q, T) and evaluate $\Pi_{\mu\nu}$ only for some of its limiting forms.

To proceed, define

$$\begin{aligned}\mathcal{S}_{\mu\nu}(\mathbf{k}, T; i\Omega, \mathbf{q}) &= \frac{1}{\beta}\sum_{\omega_n}\frac{L_{\mu\nu}(i\omega_n, \mathbf{k}; i\Omega, \mathbf{q})}{(\mathbf{k}^2-(i\omega_n)^2)((\mathbf{k}+\mathbf{q})^2-(i\omega_n+i\Omega)^2)} && \text{(B.67)}\\ &= -\oint_\Gamma \frac{dz}{2\pi i}\frac{L_{\mu\nu}(z, \mathbf{k}; i\Omega, \mathbf{q})}{(\mathbf{k}^2-z^2)((\mathbf{k}+\mathbf{q})^2-(z+i\Omega)^2)}\frac{1}{e^{\beta z}+1}, && \text{(B.68)}\end{aligned}$$

where the contour Γ includes the poles of the function $n_F(z) = 1/(e^{\beta z}+1)$ in the positive (counterclockwise) sense and the negative sign comes from the residue of $n_F(z)$ being -1. By deforming the contour around the remaining four poles it is easy to see that

$$\mathcal{S}_{\mu\nu}(\mathbf{k}, T; i\Omega, \mathbf{q}) = \frac{n_F(|\mathbf{k}|)}{2|\mathbf{k}|}\frac{L_{\mu\nu}(|\mathbf{k}|, \mathbf{k}; i\Omega, \mathbf{q})}{((i\Omega+|\mathbf{k}|)^2-(\mathbf{k}+\mathbf{q})^2)} - \frac{n_F(-|\mathbf{k}|)}{2|\mathbf{k}|}\frac{L_{\mu\nu}(-|\mathbf{k}|, \mathbf{k}; i\Omega, \mathbf{q})}{((i\Omega-|\mathbf{k}|)^2-(\mathbf{k}+\mathbf{q})^2)} + \frac{n_F(|\mathbf{k}+\mathbf{q}|)}{2|\mathbf{k}+\mathbf{q}|}\frac{L_{\mu\nu}(|\mathbf{k}+\mathbf{q}|-i\Omega, \mathbf{k}; i\Omega, \mathbf{q})}{((i\Omega-|\mathbf{k}+\mathbf{q}|)^2-\mathbf{k}^2)} - \frac{n_F(-|\mathbf{k}+\mathbf{q}|)}{2|\mathbf{k}+\mathbf{q}|}\frac{L_{\mu\nu}(-|\mathbf{k}+\mathbf{q}|-i\Omega, \mathbf{k}; i\Omega, \mathbf{q})}{((i\Omega+|\mathbf{k}+\mathbf{q}|)^2-\mathbf{k}^2)}. \quad \text{(B.69)}$$

So

$$\Pi_{\mu\nu}(\Omega, \mathbf{q}) = 4N\int\frac{d^2k}{(2\pi)^2}\mathcal{S}_{\mu\nu}(\mathbf{k}, T; i\Omega, \mathbf{q}). \quad \text{(B.70)}$$

We shall now concentrate on Π_{ij} where i, j denote the spatial indices. From here we shall be able to extract both Π_A^F and Π_B^F. Since we eventually integrate over all $\mathbf{k}$, we can shift $\mathbf{k}+\mathbf{q}\to\mathbf{k}$ in the last two terms in the Eq. (B.69) as well as reverse the direction of $\mathbf{k}$ to obtain

$$\Pi_{ij}(i\Omega, \mathbf{q}) = 4N\int\frac{d^2k}{(2\pi)^2}\frac{2n_F(|\mathbf{k}|)-1}{2|\mathbf{k}|}\left(\frac{2k_ik_j+k_iq_j+k_jq_i-\delta_{ij}(\mathbf{k}\cdot\mathbf{q}-i\Omega|\mathbf{k}|)}{(i\Omega)^2+2i\Omega|\mathbf{k}|-2\mathbf{k}\cdot\mathbf{q}-\mathbf{q}^2}+(i\Omega\to -i\Omega)\right). \quad \text{(B.71)}$$

Note that all the T dependence resides in the Fermi occupation factor n_F. Since its argument in the Eq.(B.71) is positive definite, at $T=0$, $n_F(|\mathbf{k}|)=0$ and the result is $\Pi_{\mu\nu}(q; T=0) = Nq/8(\delta_{\mu\nu}-q_\mu q_\nu/q^2)$ as obtained by dimensional regularization.

Therefore, at finite T we can write

$$\Pi_{\mu\nu}(q;T) = N\frac{q}{8}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \delta\Pi_{\mu\nu}(i\Omega, \mathbf{q}) \tag{B.72}$$

where

$$\delta\Pi_{ij}(i\Omega, \mathbf{q}) = 4N \int \frac{d^2k}{(2\pi)^2}\frac{n_F(|\mathbf{k}|)}{|\mathbf{k}|}\left(\frac{2k_ik_j + k_iq_j + k_jq_i - \delta_{ij}(\mathbf{k}\cdot\mathbf{q} - i\Omega|\mathbf{k}|)}{(i\Omega)^2 + 2i\Omega|\mathbf{k}| - 2\mathbf{k}\cdot\mathbf{q} - \mathbf{q}^2} + (i\Omega \to -i\Omega)\right). \tag{B.73}$$

We next rotate the coordinate axis so that the vector $\mathbf{q}$ is aligned with the x-axis. Since the denominator is even under reflection by the (new) x-axis we can simplify the above equality to read

$$\delta\Pi_{ij}(i\Omega, \mathbf{q}) = \frac{N}{\pi^2}\left(\delta_{ij} - \frac{q_iq_j}{\mathbf{q}^2}\right)\int_0^\infty dk n_F(k)\int_0^{2\pi} d\theta \frac{\mathcal{J}_B(i\Omega, |\mathbf{q}|, k; \cos\theta)}{\mathcal{K}(i\Omega, |\mathbf{q}|, k) - \cos\theta} +$$
$$\frac{N}{\pi^2}\frac{q_iq_j}{\mathbf{q}^2}\int_0^\infty dk n_F(k)\int_0^{2\pi} d\theta \frac{\mathcal{J}_A(i\Omega, |\mathbf{q}|, k; \cos\theta)}{\mathcal{K}(i\Omega, |\mathbf{q}|, k) - \cos\theta} + (i\Omega \to -i\Omega), \tag{B.74}$$

where

$$\begin{aligned}
\mathcal{J}_A(i\Omega, |\mathbf{q}|, k; \cos\theta) &= \frac{i\Omega}{2|\mathbf{q}|} + \frac{1}{2}\cos\theta + \frac{k}{|\mathbf{q}|}\cos^2\theta, \\
\mathcal{J}_B(i\Omega, |\mathbf{q}|, k; \cos\theta) &= \frac{i\Omega + 2k}{2|\mathbf{q}|} - \frac{1}{2}\cos\theta - \frac{k}{|\mathbf{q}|}\cos^2\theta, \\
\mathcal{K}(i\Omega, |\mathbf{q}|, k) &= \frac{(i\Omega)^2 + 2i\Omega k - \mathbf{q}^2}{2k|\mathbf{q}|}.
\end{aligned} \tag{B.75}$$

Now, $\Pi^F_{\mu\nu} = \Pi^F_A A_{\mu\nu} + \Pi^F_B B_{\mu\nu}$ (see Eq. B.56). Once written in this form, we can read off the finite T corrections to Π^F_A and Π^F_B:

$$\delta\Pi^F_A = \frac{\mathbf{q}^2 + \Omega^2}{\Omega^2}\frac{N}{\pi^2}\int_0^\infty dk n_F(k)\int_0^{2\pi} d\theta \frac{\mathcal{J}_A(i\Omega, |\mathbf{q}|, k; \cos\theta)}{\mathcal{K}(i\Omega, |\mathbf{q}|, k) - \cos\theta} + (i\Omega \to -i\Omega) \tag{B.76}$$

$$\delta\Pi^F_B = \frac{N}{\pi^2}\int_0^\infty dk n_F(k)\int_0^{2\pi} d\theta \frac{\mathcal{J}_B(i\Omega, |\mathbf{q}|, k; \cos\theta)}{\mathcal{K}(i\Omega, |\mathbf{q}|, k) - \cos\theta} + (i\Omega \to -i\Omega) \tag{B.77}$$

The above integrals can be computed analytically using contour integration since

$$\int_0^{2\pi} d\theta\, f(\cos\theta) = \oint_C \frac{dz}{iz}\, f\left(\frac{z + z^{-1}}{2}\right), \tag{B.78}$$

where the contour of integration is a unit circle around the origin encircled counterclockwise. Since eventually we will need retarded polarization function in order to compute physical quantities, we analytically continue the outside frequency

$i\Omega \to \Omega + i0^+$. Putting everything together we finally arrive at

$$\Pi^F_{A,B}(i\Omega \to \Omega + i0^+, |\mathbf{q}|, T) = \Pi^{Fret}_{A,B}(\Omega, |\mathbf{q}|, T) \tag{B.79}$$

where

$$\begin{aligned}
&\mathcal{R}e\Pi^{Fret}_A(\Omega, |\mathbf{q}|, T) = \\
&\Theta(\mathbf{q}^2-\Omega^2)\frac{|\mathbf{q}|}{8}\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}\left[1+\frac{16\ln 2}{\pi}\frac{T}{|\mathbf{q}|}\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}-\frac{4}{\pi}\int_0^{\frac{\Omega}{|\mathbf{q}|}} dx\sqrt{1-x^2}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(\frac{\Omega}{|\mathbf{q}|}-x\right)\right)-\right. \\
&\left.\frac{4}{\pi}\int_{\frac{\Omega}{|\mathbf{q}|}}^1 dx\sqrt{1-x^2}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)-\frac{4}{\pi}\int_0^1 dx\sqrt{1-x^2}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)\right]+ \\
&\Theta(\Omega^2-\mathbf{q}^2)\frac{|\mathbf{q}|}{8}\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}\left[-\frac{16\ln 2}{\pi}\frac{T}{|\mathbf{q}|}\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}+\frac{4}{\pi}\int_1^{\frac{\Omega}{|\mathbf{q}|}} dx\sqrt{x^2-1}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(\frac{\Omega}{|\mathbf{q}|}-x\right)\right)+\right. \\
&\left.\frac{4}{\pi}\int_{\frac{\Omega}{|\mathbf{q}|}}^\infty dx\sqrt{x^2-1}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)-\frac{4}{\pi}\int_1^\infty dx\sqrt{x^2-1}\, n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)\right]
\end{aligned} \tag{B.80}$$

and

$$\begin{aligned}
&\mathcal{I}m\Pi^{Fret}_A(\Omega, |\mathbf{q}|, T) = \Theta(\mathbf{q}^2-\Omega^2)\frac{|\mathbf{q}|}{8}\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}\times \\
&\quad\left[\frac{4}{\pi}\int_1^\infty dx\sqrt{x^2-1}\,\left\{n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)-n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)\right\}\right]+ \\
&\quad sgn\Omega\;\; \Theta(\Omega^2-\mathbf{q}^2)\frac{|\mathbf{q}|}{8}\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}\left[\frac{4}{\pi}\int_{-1}^1 dx\sqrt{1-x^2}\,\left\{n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)-\frac{1}{2}\right\}\right]
\end{aligned} \tag{B.81}$$

where $\Theta(x)$ is a Heaviside step function. Similarly,

$$
\begin{aligned}
&\mathcal{R}e\Pi_B^{Fret}(\Omega, |\mathbf{q}|, T) = \\
&\Theta(\mathbf{q}^2-\Omega^2)\frac{|\mathbf{q}|}{8}\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}\left[1+\frac{16\ln 2}{\pi}\frac{T}{|\mathbf{q}|}\frac{\frac{\Omega^2}{\mathbf{q}^2}}{\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}}-\frac{4}{\pi}\int_0^{\frac{\Omega}{\mathbf{q}}}dx\frac{x^2}{\sqrt{1-x^2}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(\frac{\Omega}{|\mathbf{q}|}-x\right)\right)-\right.\\
&\left.\frac{4}{\pi}\int_{\frac{\Omega}{\mathbf{q}}}^1 dx\frac{x^2}{\sqrt{1-x^2}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)-\frac{4}{\pi}\int_0^1 dx\frac{x^2}{\sqrt{1-x^2}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)\right]+\\
&\Theta(\Omega^2-\mathbf{q}^2)\frac{|\mathbf{q}|}{8}\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}\left[\frac{16\ln 2}{\pi}\frac{T}{|\mathbf{q}|}\frac{\frac{\Omega^2}{\mathbf{q}^2}}{\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}}-\frac{4}{\pi}\int_1^{\frac{\Omega}{\mathbf{q}}}dx\frac{x^2}{\sqrt{x^2-1}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(\frac{\Omega}{|\mathbf{q}|}-x\right)\right)-\right.\\
&\left.\frac{4}{\pi}\int_{\frac{\Omega}{\mathbf{q}}}^\infty dx\frac{x^2}{\sqrt{x^2-1}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)+\frac{4}{\pi}\int_1^\infty dx\frac{x^2}{\sqrt{x^2-1}}\,n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)\right]
\end{aligned}
\tag{B.82}
$$

and

$$
\begin{aligned}
\mathcal{I}m\Pi_B^{Fret}(\Omega, |\mathbf{q}|, T) &= \Theta(\mathbf{q}^2-\Omega^2)\frac{|\mathbf{q}|}{8}\sqrt{1-\frac{\Omega^2}{\mathbf{q}^2}}\times\\
&\left[\frac{4}{\pi}\int_1^\infty dx\frac{x^2}{\sqrt{x^2-1}}\,\left\{n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)-n_F\left(\frac{|\mathbf{q}|}{2T}\left(x-\frac{\Omega}{|\mathbf{q}|}\right)\right)\right\}\right]+\\
&sgn\Omega\ \ \Theta(\Omega^2-\mathbf{q}^2)\frac{|\mathbf{q}|}{8}\sqrt{\frac{\Omega^2}{\mathbf{q}^2}-1}\left[\frac{4}{\pi}\int_{-1}^1 dx\frac{x^2}{\sqrt{1-x^2}}\,\left\{n_F\left(\frac{|\mathbf{q}|}{2T}\left(x+\frac{\Omega}{|\mathbf{q}|}\right)\right)-\frac{1}{2}\right\}\right]
\end{aligned}
\tag{B.83}
$$

The expressions for the imaginary parts of Π_A^{Fret} and Π_B^{Fret} (B.81) and (B.83) can be further simplified using an infinite series of Bessel functions, but there is no further simplification of the real parts of Π_A^{Fret} and Π_B^{Fret}. Nevertheless, there is an important fact that ought to be stressed: the expressions (B.80-B.83) reflect the quantum critical scaling of the theory near its $T = 0$ fixed point. As can be seen from above, we can write the following scaling relations:

$$
\Pi_{A,B}^{Fret}(\Omega, |\mathbf{q}|, T) = NT\ \mathcal{P}_{A,B}^F\left(\frac{|\mathbf{q}|}{T}, \frac{\Omega}{T}\right) \tag{B.84}
$$

In the limiting cases of small or large argument, both real and imaginary part of the function $\mathcal{P}_{A,B}^F\left(\frac{q}{T}, \frac{\Omega}{T}\right)$ can be evaluated analytically.

B.6 Free energy scaling of QED_3

We shall now use the results of the previous section to compute the T dependence of the thermodynamic free energy from which we can find the electronic contribution to the specific heat. Since to order $1/N$ the effective action for the gauge field is quadratic, we can perform the functional integral over its two "modes" exactly. Thus,

$$F = \frac{1}{\beta}\ln(Det(\Pi_A)) + \frac{1}{\beta}\ln(Det(\Pi_B)), \tag{B.85}$$

where $\Pi_A = \frac{\mathbf{q}^2}{\mathbf{q}^2+\Omega^2}(\Pi_A^0 + \Pi_A^F)$ and $\Pi_B = \Pi_B^0 + \Pi_B^F$. In addition we have,

$$F = \frac{1}{\beta}\sum_{\Omega_n}\int\frac{d^2q}{(2\pi)^2}\big(\ln\left[\Pi_A(i\Omega_n,\mathbf{q})\right] + \ln\left[\Pi_B(i\Omega_n,\mathbf{q})\right]\big). \tag{B.86}$$

Now, $\ln(z)$ is analytic in the upper complex plane with a branch-cut along the x axis. Recognizing this fact, we can trade the summation over the bosonic Matsubara frequencies $\Omega_n = 2\pi nT$ for an integral along the branch cut:

$$\begin{aligned}
\frac{1}{\beta}\sum_{\Omega_n}\ln\left[\Pi_{A,B}(i\Omega_n,\mathbf{q})\right] &= \oint_\Gamma \frac{dz}{2\pi i}\frac{\ln\left[\Pi_{A,B}(z,\mathbf{q})\right]}{e^{\beta z}-1}\\
&= \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi i}\frac{\ln\left[\Pi_{A,B}(\Omega+i0^+,\mathbf{q})\right]-\ln\left[\Pi_{A,B}(\Omega-i0^+,\mathbf{q})\right]}{e^{\beta\Omega}-1}\\
&= 2\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\frac{\mathcal{I}m\ln\left[\Pi_{A,B}(\Omega+i0^+,\mathbf{q})\right]}{e^{\beta\Omega}-1}\\
&= 2\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\frac{1}{e^{\beta\Omega}-1}\tan^{-1}\left(\frac{\mathcal{I}m\Pi_{A,B}^{ret}(\Omega,\mathbf{q})}{\mathcal{R}e\Pi_{A,B}^{ret}(\Omega,\mathbf{q})}\right)
\end{aligned} \tag{B.87}$$

Now, the crucial observation about the polarization functions is that they can be rescaled as

$$\begin{aligned}
\Pi_{A,B}^{Fret}(\Omega,|\mathbf{q}|,T) &= NT\mathcal{P}_{A,B}^F\left(\frac{|\mathbf{q}|}{T},\frac{\Omega}{T}\right)\\
\Pi_{A,B}^{0ret}(\Omega,|\mathbf{q}|,T) &= \frac{T^2}{e^2}\mathcal{P}_{A,B}^0\left(\frac{|\mathbf{q}|}{T},\frac{\Omega}{T}\right).
\end{aligned} \tag{B.88}$$

This can be used to argue that the temperature dependence of free energy reads

$$F = -NT^3\frac{3\zeta(3)}{4\pi} + T^3\Phi\left(\frac{T}{Ne^2}\right), \tag{B.89}$$

where the fist term is the contribution from N free 4−component Dirac fermions and the second term comes from the interaction with the gauge field. By numerical integrations we have found out that the function $\Phi(x)$ is regular at x=0 (there are two log singularities coming from Π_A and Π_B which cancel). This result shows that there is no anomalous dimension in the free energy i.e. that the $z = 1$ hyperscaling at the QED_3 quantum critical point holds. Therefore, the leading scaling of the specific heat in finite T QED_3 is $c_v \sim T^2$.

Bibliography

[1] Bednorz and Muller, Z. Phys. B **64**, 189 (1986).

[2] P. W. Anderson, *Science* **235**, 1196 (1987).

[3] J. Orenstein and A. J. Physics, *Science* **288**, 468 (2000).

[4] A. Damascelli, Z. Hussain, and Zhi-Xun Shen, Rev. Mod. Phys.**75**, 473 (2003).

[5] C. C. Tsuei and J. R. Kirtley, Rev. Mod. Phys.**72**, 969 (2000).

[6] W. N. Hardy *et. al.*, Phys. Rev. Lett. **70**, 3999 (1993).

[7] M. Chiao *et. al.*, Phys. Rev. **B62**, 3554 (2000).

[8] Z. A. Xu *et al.*, Nature **406**, 486 (2000); Y. Wang *et al.*, cond-mat/0108242.

[9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957); J.R. Schrieffer, *Theory of superconductivity* (New York, W.A. Benjamin, 1964.)

[10] D. J. Van Harlingen, Rev. Mod. Phys. **67**, 515 (1995).

[11] P. A. Lee, Science **277**, 50 (1997).

[12] See e.g. W.A. Atkinson, P.J. Hirschfeld, A.H. MacDonald, K. Ziegler, cond-mat/0005487 and references therein.

[13] G. E. Volovik, Sov. Phys. JETP Lett. **58**, 469 (1993).

[14] S. K. Yip and J. A. Sauls, Phys. Rev. Lett. **69**, 2264 (1992).

[15] S. H. Simon and P. A. Lee, Phys. Rev. Lett. **78**, 1548 (1997).

[16] T. M. Rice, Nature **396**, 627 (1998); K. Ishida, H. Mukuda, Y. Kitaoka, et. al. Nature **396**, 658 (1998).

[17] G. E. Volovik, cond-mat/9709159; Sov. Phys. JETP Lett. **70**, 609 (1999).

[18] M. Matsumoto and M. Sigrist, J. Phys. Soc. Jpn., **68** (3), 724 (1999); Physica B **281**, 973 (2000).

[19] M. Franz and A. J. Millis, Phys. Rev. B **58**, 14572 (1998).

[20] H.-J. Kwon and A. T. Dorsey, Phys. Rev. **B60**, 6438 (1999).

[21] L. Balents, M. P. A. Fisher and C. Nayak, Phys. Rev. **B60**, 1654 (1999).

[22] M. Franz and Z. Tešanović, Phys. Rev. Lett. **84**, 554 (2000); O. Vafek *et al.*, Phys. Rev. **B63**, 134509 (2001).

[23] C. Caroli, P.G. de Gennes, J. Matricon, Phys. Lett. **9**, 307 (1964).

[24] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, Reading, MA, 1989).

[25] M. Franz and Z. Tešanović, Phys. Rev. Lett. **80**, 4763 (1998).

[26] X. R. Resende and A. H. MacDonald, Bull. Am. Phys. Soc. **43**, 573, 1998.

[27] Y. Wang and A. H. MacDonald, Phys. Rev. **B52**, R3876 (1995).

[28] K. Yasui and T. Kita, Phys. Rev. Lett. **83**, 4168 (1999).

[29] L. P. Gor'kov and J. R. Schrieffer, Phys. Rev. Lett. **80**, 3360 (1998).

[30] P. W. Anderson, cond-mat/9812063.

[31] A. S. Melnikov, J. Phys. Cond. Matt. **82**, 4703 (1999).

[32] L. Marinelli, B. I. Halperin and S. H. Simon, Phys. Rev. **B62** 3488 (2000).

[33] A. Moroz, Phys. Rev. A **53**, 669 (1996).

[34] J. Ye, Phys. Rev. Lett. **86**, 316 (2001).

[35] M. Nielsen and P. Hedegård, Phys. Rev. B **51**, 7679 (1995).

[36] T. Senthil, M. P. A. Fisher, L. Balents, and C. Nayak, cond-mat/9808001.

[37] Quantized spin currents in superfluids and superconductors in different physical situations have also been considered in G. E. Volovik, JETP Lett. **66**, 522 (1997) and G. E. Volovik and V. M. Yakovenko, J. Phys. Cond. Matt. **1** 5263, (1989). See also V. M. Yakovenko, Phys. Rev. Lett. **65**, 251 (1990).

[38] A. C. Durst and P. A. Lee, Phys. Rev. **B62**, 1270 (2000).

[39] A. Vishwanath, Phys. Rev. Lett. **87**, 217004 (2001), A. Vishwanath Phys. Rev. **B66**, 064504 (2002).

[40] This quantization is an example of a general class of problems exhibiting the "parity anomaly" as originally discussed in F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988). For application to spin currents in chiral magnets see F. D. M. Haldane and D. P. Arovas, Phys. Rev. **B52**, 4223 (1995). For various manifestations of such quantized transport in superfluids and superconductors see Refs. [36, 37].

[41] For translationally invariant superconductors with broken time-reversal and parity such topological invariant has been derived by N. Read and Dmitry Green, Phys. Rev. **B61** 10267, (2000).

[42] In a typical situation, the combined symmetry of BdG equations and the inversion symmetry of a vortex lattice conspire to restrict σ^s_{xy} to *even* multiples of $\hbar/8\pi$.

[43] For discussion of some general properties of the BCS-Hofstadter problem in d-wave superconductors see Y. Morita and Y. Hatsugai, cond-mat/0007067.

[44] In real superconductors, this sequence of transitions will also be influenced by strong disorder. For spin QHE in dirty superconductors with broken orbital time-reversal symmetry see I. A. Gruzberg, A. W. Ludwig, and N. Read, Phys. Rev. Lett. **82** 4524 (1999); V. Kagalovsky, B. Horovitz, Y. Avishai, and J. T. Chalker, Phys. Rev. Lett. **82** 3516, (1999). In our case, perhaps the most interesting and relevant form of disorder is the one associated with disorder in vortex positions.

[45] O. Vafek, A. Melikyan, M. Franz, Z. Tešanović, Phys. Rev. **B63**, 134509 (2001).

[46] J. von Neumann and E. Wigner, *Physik. Z.*, **30** 467 (1927).

[47] T. Senthil, J. B. Marston, and M. P. A. Fisher, cond-mat/9902062.

[48] J. M. Luttinger, Phys. Rev. **135**, A1505 (1964).

[49] Yu. N. Obraztsov, Fiz. Tverd. tela **6**, 414 (1964) [Sov. Phys.-Solid State **6**, 331 (1964)]; **7**, 455 (1965).

[50] L. Smrčka and P. Středa, J. Phys. C **10** 2153 (1977).

[51] H. Oji and P. Streda, Phys. Rev. **B31**, 7291 (1985).

[52] G. D. Mahan, *Many-Particle Physics* (Plenum, New York, 1990).

[53] M. Kohmoto, Annals of Physics **160**, 343 (1985).

[54] M. Jonson and S. M. Girvin, Phys. Rev. **B29**, 1939 (1984).

[55] C. Poople *et. al.*, *Superconductivity* (Academic Press, San Diego, 1995).

[56] V. Ambegaokar and A. Griffin, Phys. Rev. **B137**, A1151 (1965).

[57] J. Corson *et al.*, Nature **398**, 221 (1999).

[58] M. Franz and Z. Tešanović, Phys. Rev. Lett. **87**, 257003 (2001); M. Franz, Z. Tesanovic and O. Vafek, Phys. Rev. **B66**, 054535 (2002).

[59] R. D. Pisarski, Phys. Rev. **D29**, 2423 (1984); T. W. Appelquist *et al.*, Phys. Rev. **D33**, 3704 (1986); M. R. Pennington and D. Walsh, Phys. Lett. B **253**, 246 (1991); I. J. R. Aitchison and N. E. Mavromatos, Phys. Rev. **B53**, 9321 (1996); P. Maris, Phys. Rev. **D54**, 4049 (1996).

[60] T. W. Appelquist, A. G. Cohen, Schmaltz Phys. Rev. **D60** 045003 (1999); Hands et. al. hep-lat/0209133.

[61] P. W. Anderson, *The theory of superconductivity in the high-T_c cuprates*, Princeton University Press (1997).

[62] B. W. Hoogenboom *et al.*, Phys. Rev. **B62**, 9179 (2000).

[63] T. Nakano *et al.*, Journal J. Phys. Soc. **67**, 2622 (1998).

[64] The labels "compact" and "non-compact" gauge fields pasted on a_μ and v_μ have the following precise meaning: let us for a moment forget about definitions (3.7,3.8) and simply treat a_μ and v_μ appearing in (3.6) as two completely *independent* gauge fields. We further introduce the *compactified* gauge transformations (for example, see I. I. Kogan and A. Kovner, Phys. Rev. **D53**, 4510 (1996); *ibid* **51**, 1948 (1995)) defined as:

$$a_\mu \to a_\mu + \partial_\mu \theta_{\rm reg} + \partial_\mu \theta_{\rm sing}$$

where $\theta_{\rm reg}(x)$ describes the ordinary smooth or regular gauge transformations. These familiar gauge transformations are extended to include singular gauge transformations described by $\theta_{\rm sing}(x)$, itself determined from:

$$\left(\partial \times \partial \theta_{\rm sing}(x)\right)_\mu = 2\pi \sum_{k=1}^{\mathcal{N}} \int_L dx_{k\mu} \delta(x - x_k) \ ,$$

where the sum runs over an arbitrary number $\mathcal{N}$ of unit-flux Dirac strings and $\int_L$ denotes a line integral. We now inquire whether the theory (3.6) is invariant under such compactified gauge transformations when applied to a_μ. The answer is "yes". However, when the same type of transformation is separately applied to v_μ, the answer is "no", due to its Meissner coupling to BdG fermions. Hence the labels. Note that both $v_{A\mu}$ and $v_{B\mu}$ are non-compact in the same sense.

[65] For computational convenience we define $\mathbf{V}_s$ as twice the conventional superfluid density; see Ref. [66].

[66] M. Tinkham, *Introduction to Superconductivity*, Krieger Publishing Company (1975).

[67] An exception to this general situation is the Anderson gauge, $\varphi_A = \varphi$, $\varphi_B = 0$ or $\varphi_A = 0$, $\varphi_B = \varphi$ (P. W. Anderson, cond-mat/9812063), which has least desirable properties in this regard, resulting in $\mathbf{v} = \mathbf{a}$ ($v_\mu = a_\mu$), or an *infinitely* strong coupling between $\mathbf{v}$ and $\mathbf{a}$. It is then impossible to decouple $\mathbf{v}$ (v_μ) and $\mathbf{a}$ (a_μ), even in the RG sense. Consequently, any attempt to construct the effective theory by using the Anderson gauge is unlikely to be successful. To illustrate this, note that in the Anderson gauge (also in a general "$A-B$" gauge) the system has *finite* overall magnetization: $M = n_\uparrow - n_\downarrow = \langle\bar{\psi}_\uparrow\psi_\uparrow\rangle - \langle\bar{\psi}_\downarrow\psi_\downarrow\rangle = \langle\tilde{\psi}_\uparrow^\dagger\tilde{\psi}_\uparrow\rangle - \langle\tilde{\psi}_\downarrow^\dagger\tilde{\psi}_\downarrow\rangle \neq 0$! In contrast, in the FT gauge (3.4,3.11) $M = 0$ as it must be. Again, this preference for the FT gauge is a consequence of the intricate nature of the "coarse-graining" process – if the gauge fields are defined on *discrete* vortex configurations $\{\mathbf{r}_i(\tau)\}$ (3.7,3.8) the magnetization vanishes with any choice of gauge.

[68] T. Senthil and M. P. A. Fisher, Phys. Rev. **B62**, 7850 (2000).

[69] D. A. Bonn *et al.*, Nature **414**, 887 (2001).

[70] Z. Tešanović, Phys. Rev. **B59**, 6449 (1999); see Appendices A and B.

[71] L. Balents, M. P. A. Fisher and C. Nayak, Phys. Rev. **B60**, 1654 (1999).

[72] A. Nguyen and A. Sudbø, Phys. Rev. **B60**, 15307 (1999).

[73] S. H. Simon and P. A. Lee, Phys. Rev. Lett. **78**, 1548 (1997).

[74] V. Gusynin, A. Hams and M. Reenders, Phys. Rev. **D63**, 045025 (2001).

[75] M. Reenders, cond-mat/0110168.

[76] See e.g. M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley 1995.

[77] X. -G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996); P. A. Lee, N. Nagaosa, T. K. Ng, and X. -G. Wen, Phys. Rev. **B57**, 6003 (1998).

[78] T. W. Appelquist and U. Heinz, Phys. Rev. **D24**, 2169 (1981); D. Atkinson, P. W. Johnson and P. Maris, Phys. Rev. **D42**, 602 (1990).

[79] M. R. Pennington and D. Walsh, Phys. Lett B **253**, 246 (1991).

[80] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **76**, 4588 (1996).

[81] D. H. Kim, P. A. Lee and X.-G. Wen, Phys. Rev. Lett. **79**, 2109 (1997); D. H. Kim and P. A. Lee, Ann. Phys. **272**, 130 (1999).

[82] Jean Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, 1996.

[83] Z. Tesanovic, O. Vafek and M. Franz Phys. Rev. **B65**, 180511(R) (2002); see also I.F. Herbut, Phys. Rev. Lett. **88**, 047006 (2002).

[84] T. Appelquist *et al.*, Phys. Rev. **D33**, 3774 (1986).

[85] See e.g. M. R. Norman, A. Kaminski, J. Mesot and J. C. Campuzano, Phys. Rev. **B63**, 140508 (2001).

[86] H. Kleinert, *Gauge Fields in Condensed Matter*, vol. I, World Scientific, Singapore (1989).

Vita

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